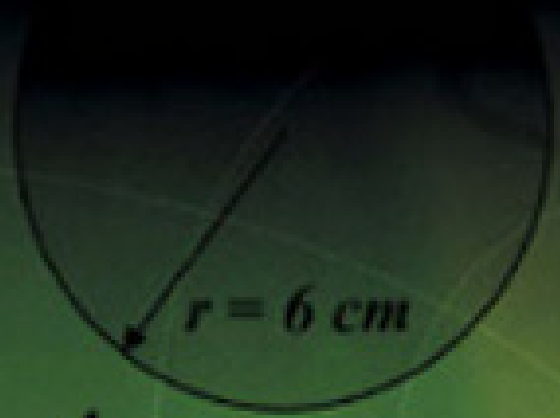
CGP Education

California Mathematics Course One



Student Textbook

California Standards-Driven Program

California

Mathematics

Course One

Student Textbook

California Standards-Driven Program



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This Textbook provides comprehensive coverage of all the California grade 6 Standards. The Textbook is divided into seven Chapters. Each of the Chapters is broken down into small, manageable Lessons and each Lesson covers a specific Standard or part of a Standard.

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California Grade Six Mathematics Standards

The following table lists all the California Mathematics Content Standards for grade 6 with cross references to where each Standard is covered in this Textbook. Each Lesson begins by quoting the relevant Standard in full, together with a clear and understandable objective. This will enable you to measure your progression against the California grade 6 Standards as you work your way through the Program.

California Standard		Number Sense	
1.0	★	Students compare and order positive and negative fractions, decimals, and mixed numbers. Students solve problems involving fractions, ratios, proportions, and percentages:	**Chapters 1, 3, 4**
1.1	★	Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.	**Chapters 1, 3**
1.2	★	Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b, a to b, $a:b$).	**Chapter 4**
1.3	★	Use proportions to solve problems (e.g., determine the value of N if $\frac{4}{7}=\frac{N}{21}$, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.	**Chapter 4**
1.4	★	Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.	**Chapter 3**
2.0	★	Students calculate and solve problems involving addition, subtraction, multiplication, and division:	**Chapters 1, 3**
2.1		Solve problems involving addition, subtraction, multiplication, and division of positive fractions and explain why a particular operation was used for a given situation.	**Chapter 3**
2.2		Explain the meaning of multiplication and division of positive fractions and perform the calculations (e.g., $\frac{5}{8}\div\frac{15}{16}=\frac{5}{8}\times\frac{16}{15}=\frac{2}{3}$).	**Chapter 3**
2.3	★	Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations, that use positive and negative integers and combinations of these operations.	**Chapter 1**
2.4	★	Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).	**Chapter 3**

California Standard		Algebra and Functions	
1.0		Students write verbal expressions and sentences as algebraic expressions and equations; they evaluate algebraic expressions, solve simple linear equations, and graph and interpret their results:	**Chapter 2**
1.1	★	Write and solve one-step linear equations in one variable.	**Chapter 2**
1.2		Write and evaluate an algebraic expression for a given situation, using up to three variables.	**Chapter 2**
1.3		Apply algebraic order of operations and the commutative, associative, and distributive properties to evaluate expressions; and justify each step in the process.	**Chapter 2**
1.4		Solve problems manually by using the correct order of operations or by using a scientific calculator.	**Chapter 2**
2.0		Students analyze and use tables, graphs, and rules to solve problems involving rates and proportions:	**Chapter 4**
2.1		Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).	**Chapter 4**
2.2	★	Demonstrate an understanding that *rate* is a measure of one quantity per unit value of another quantity.	**Chapter 4**
2.3		Solve problems involving rates, average speed, distance, and time.	**Chapter 4**

California Grade Six Mathematics Standards

3.0	Students investigate geometric patterns and describe them algebraically:	**Chapter 2, 7**
3.1	Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ — the formulas for the perimeter of a rectangle, the area of a triangle, and the circumference of a circle, respectively).	**Chapter 2, 7**
3.2	Express in symbolic form simple relationships arising from geometry.	**Chapter 2, 7**

California Standard

Measurement and Geometry

1.0	Students deepen their understanding of the measurement of plane and solid shapes and use this understanding to solve problems:	**Chapter 7**
1.1	★ Understand the concept of a constant such as π; know the formulas for the circumference and area of a circle.	**Chapter 7**
1.2	Know common estimates of π (3.14; $\frac{22}{7}$) and use these values to estimate and calculate the circumference and the area of circles; compare with actual measurements.	**Chapter 7**
1.3	Know and use the formulas for the volume of triangular prisms and cylinders (area of base × height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.	**Chapter 7**
2.0	Students identify and describe the properties of two-dimensional figures:	**Chapter 7**
2.1	Identify angles as vertical, adjacent, complementary, or supplementary and provide descriptions of these terms.	**Chapter 7**
2.2	★ Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.	**Chapter 7**
2.3	Draw quadrilaterals and triangles from given information about them (e.g., a quadrilateral having equal sides but no right angles, a right isosceles triangle).	**Chapter 7**

California Standard

Statistics, Data Analysis and Probability

1.0	Students compute and analyze statistical measurements for data sets:	**Chapter 5**
1.1	Compute the range, mean, median, and mode of data sets.	**Chapter 5**
1.2	Understand how additional data added to data sets may affect these computations.	**Chapter 5**
1.3	Understand how the inclusion or exclusion of outliers affects these computations.	**Chapter 5**
1.4	Know why a specific measure of central tendency (mean, median) provides the most useful information in a given context.	**Chapter 5**
2.0	Students use data samples of a population and describe the characteristics and limitations of the samples:	**Chapter 5**
2.1	Compare different samples of a population with the data from the entire population and identify a situation in which it makes sense to use a sample.	**Chapter 5**
2.2	★ Identify different ways of selecting a sample (e.g., convenience sampling, responses to a survey, random sampling) and which method makes a sample more representative for a population.	**Chapter 5**
2.3	★ Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.	**Chapter 5**

2.4	★	Identify data that represent sampling errors and explain why the sample (and the display) might be biased.	**Chapter 5**
2.5	★	Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.	**Chapter 5**
3.0		Students determine theoretical and experimental probabilities and use these to make predictions about events:	**Chapter 6**
3.1	★	Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.	**Chapter 6**
3.2		Use data to estimate the probability of future events (e.g., batting averages or number of accidents per mile driven).	**Chapter 6**
3.3	★	Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, $1 - P$ is the probability of an event not occurring.	**Chapter 6**
3.4		Understand that the probability of either of two disjoint events occurring is the sum of the two individual probabilities and that the probability of one event following another, in independent trials, is the product of the two probabilities.	**Chapter 6**
3.5	★	Understand the difference between independent and dependent events.	**Chapter 6**

California Standard

Mathematical Reasoning

1.0		Students make decisions about how to approach problems:	**Illustrated in Chapters 1, 2, and 7, and throughout Program**
1.1		Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, identifying missing information, sequencing and prioritizing information, and observing patterns.	
1.2		Formulate and justify mathematical conjectures based on a general description of the mathematical question or problem posed.	
1.3		Determine when and how to break a problem into simpler parts.	
2.0		Students use strategies, skills, and concepts in finding solutions:	
2.1		Use estimation to verify the reasonableness of calculated results.	
2.2		Apply strategies and results from simpler problems to more complex problems.	
2.3		Estimate unknown quantities graphically and solve for them by using logical reasoning and arithmetic and algebraic techniques.	
2.4		Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.	
2.5		Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.	
2.6		Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.	
2.7		Make precise calculations and check the validity of the results from the context of the problem.	
3.0		Students move beyond a particular problem by generalizing to other situations:	
3.1		Evaluate the reasonableness of the solution in the context of the original situation.	
3.2		Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.	
3.3		Develop generalizations of the results obtained and the strategies used and apply them in new problem situations.	

★ Shows **key standards**

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Chapter 1

Ordering and Manipulating Numbers

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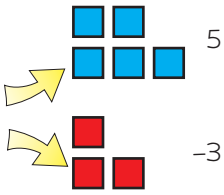
Section 1.1 introduction — an exploration into:

Zero Pairs

Lots of math involves adding and subtracting with positive and negative numbers. Positive and negative numbers "cancel each other out" — for example, if you add the numbers 1 and –1, you get zero. You can use this fact to add and subtract other numbers too.

You can represent positive and negative integers using blue and red tiles.

Each **blue tile** represents **1**. So the number **5** would be shown using **5 blue tiles**.
Each **red tile** represents **–1**. So the number **–3** would be shown using **3 red tiles**.



A red and a blue tile together make **zero**. This is called a **zero pair**.



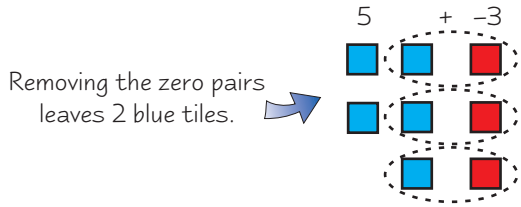
Because zero pairs have a **value** of **zero**, they **do not affect** the value of other groups of tiles.

Example

Add 5 + (–3) with integer tiles.

Solution

Take **5 blue tiles**, and **add 3 red tiles**.
You can remove the zero pairs.
This leaves **2 blue tiles**. So **5 + (–3) = 2**.



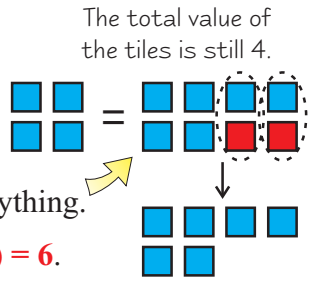
Example

Subtract 4 – (–2) with integer tiles.

Solution

Take **4 blue tiles**. To **subtract –2**, you need to **remove 2 red tiles**.
But there are no red tiles.
However, you can **add two zero pairs** — zero pairs **do not change** anything.

Now you can **remove 2 red tiles**. This leaves **6 blue tiles**. So **4 – (–2) = 6**.



Exercises

1. Use integer tiles to model and solve each of the following:
 a. –6 + 5 **b.** –2 + (–7) **c.** 8 + (–3) **d.** 5 – 7 **e.** –3 – 2 **f.** 2 – (–3)
2. How can **4 – (–6)** be solved using red and blue tiles?
 Show how removing 6 red tiles has the same effect as adding 4 blue tiles.

Round Up

You can use tiles to represent lots of addition and subtraction problems.
To do an addition, you need to add tiles. To do a subtraction, you need to remove tiles.
But remember, you can always add or remove a zero pair without affecting anything.

Lesson 1.1.1

Section 1.1
Comparing Integers

This Lesson is about some of the different kinds of numbers you need to use, and how you can use the number line to put them in order. The number line shows whether numbers are greater than or less than each other.

California Standard:

Number Sense 1.1

Compare and order positive and negative fractions, decimals, and mixed numbers **and place them on a number line**.

What it means for you:

You'll be able to arrange positive and negative numbers on the number line, and you'll learn about different types of numbers.

Key words:

- natural number
- whole number
- integer
- origin
- number line
- greater than
- less than

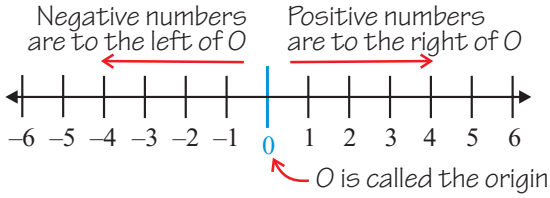
Don't forget:

0 isn't positive or negative.

The Number Line Shows the Order of Numbers

The **number line** is a useful diagram that shows how numbers are ordered.

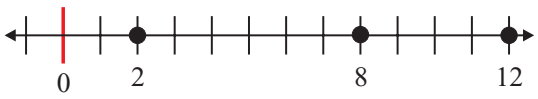
Numbers to the right of 0 are **positive** and numbers to the left of 0 are **negative**.



Example 1

Arrange the numbers 2, 8, and 12 on the number line.

Solution



Just count to the **right** of 0 by 2, 8, and 12 steps.

You always count steps from 0, but if it's a **negative** number then remember to count to the **left**.

Don't forget:

Negative and positive versions of the same number are the same distance from the origin. For example, 3 and –3 are both 3 steps away from 0 on the number line, but in different directions. The negative side of the number line is like a mirror image of the positive side.

Example 2

Arrange the numbers –3, –5, 3, and 7 on the number line.

Solution



Remember that –3 and –5 are to the **left** of 0.

Guided Practice

1. Arrange the numbers 3, 4, –1, and 8 on the number line.

Say which side of the origin the numbers in Exercises 2–5 would go on.

2. –23
3. 9
4. 6
5. –1

The Number Line Lets You Compare Numbers

Check it out:
The further to the right on a number line, the greater a number is. It doesn't matter which side of the origin the numbers are.

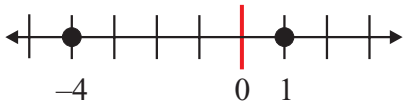
If two numbers are plotted on the number line, then **the number to the left is less than the number to the right**. You could also say that **the number to the right is greater than the number to the left**.

There are two important symbols used to order and compare two numbers. The symbol **>** means "**greater than**" and the symbol **<** means "**less than**."

Example 3

By placing them on the number line, show that 1 is greater than –4.

Solution



1 is to the right of –4, so **1 > –4**.

Check it out:
–4 is less than –2, even though 4 is greater than 2. That's because –4 goes on the left of –2 on the number line.

Guided Practice

Write these numbers with the correct < or > symbol, then use the number line to say which number in each Exercise is greater.

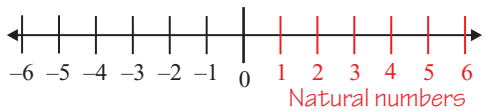
6. 7, 3
7. 2, 3
8. –1, 10
9. –8, 0
10. –9, –11
11. –1, 1

Whole and Natural Numbers Are Types of Integers

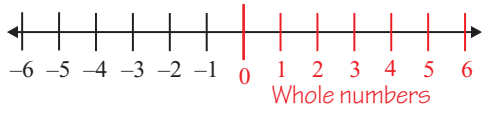
You've already been dealing with a few different **types of numbers** on the number line — now it's time to learn their names.

Check it out:
If you're just counting objects, you use natural numbers to say how many you have.

Natural numbers are also called **counting numbers** because they are the numbers used to count with — 1, 2, 3, and so on.

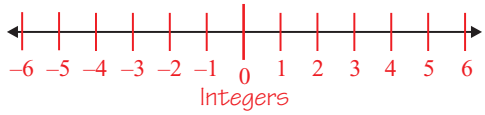


Whole numbers are like the natural numbers except that they **include the number zero**.



Check it out:
For example, –3, –2, –1, 0, 1, 2, and 3 are all integers but –0.5 is not an integer.

Integers include all the **natural numbers**, the negative versions of them, and zero.



Check it out:
Natural numbers and whole numbers have something in common — **numbers less than zero are not included**.

Don't forget:
0 is a whole number but not a natural number.

Example 4

Describe the following numbers as natural, whole, or integer. There may be more than one answer for each.

1. 1 **2.** 0 **3.** –3

Solution

1. 1 is greater than zero so it is a **natural** number.
All natural numbers are also **whole numbers** and **integers**.

2. 0 is **not a natural number**, but it is a **whole number**.
As it is a whole number it is also an **integer**.

3. –3 is less than zero so it is **not a natural number** and **not a whole number**. But it is an **integer**.

Guided Practice

Describe each number as a natural number, a whole number, or an integer. There may be more than one answer for each number.

12. 7 **13.** –9
14. –400 **15.** 1
16. 55 **17.** 0
18. 26 **19.** –16
20. –32 **21.** 3.2

Independent Practice

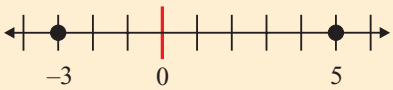
Now try these:
Lesson 1.1.1 additional questions — p428

1. Place 3, 1, 5, and –2 on a number line.

Answer each of Exercises 2–9 as true or false.
2. $4 < 9$ **3.** $23 > -40$ **4.** $0 < -1$
5. $-1 > -2$ **6.** $4 > -5$ **7.** 0 is a positive number.
8. Whole numbers can be negative.
9. All integers are whole numbers.

10. Hector is thinking of a number that is a whole number but is not a natural number. What number is Hector thinking of?

Exercises 11–13 make use of the number line shown below.
11. How many natural numbers are between –3 and 5?
12. How many whole numbers are between –3 and 5?
13. How many integers are between –3 and 5?



Round Up

Thinking in terms of the number line is a good way to compare integers so you can say which is greater than the other. It also helps you to understand what happens when you add positive and negative numbers, which is all part of the next Lesson.

Lesson 1.1.2

Adding and Subtracting Integers

California Standard:

Number Sense 2.3

Solve addition, subtraction, multiplication, and division **problems, including those arising in concrete situations, that use positive and negative integers** and combinations of these operations.

What it means for you:

You'll learn to add positive and negative numbers to integers.

Key words:

- positive
- negative
- integer

Don't forget:

Integers include all the natural numbers, the negative versions of them, and zero. So –2, –1, 0, 1, and 2 are all integers.

You've met positive and negative numbers, and you know how to arrange them on the number line. This Lesson covers addition and subtraction of integers, including negative numbers.

Adding a Positive Moves Right on the Number Line

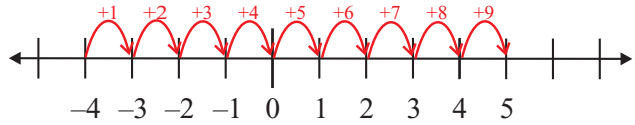
In the last Lesson, you saw that numbers on the number line are **greater** the farther to the **right** they are. When you **add** a positive integer to any number, you're making it **greater**, so you move to the **right**.

To add a positive integer, **move that number of positions to the right** on the number line. It doesn't make any difference whether the original number is positive or negative — you **always** move to the right.

Example 1

Using the number line, add 9 to –4.

Solution



Counting **9 spaces to the right of –4** on the number line shows that the answer is **5**.

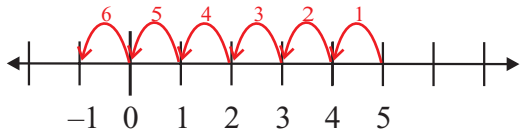
Adding a Negative Moves Left on the Number Line

When you add a negative integer, **move that number of positions to the left** on the number line.

Example 2

Using the number line, add –6 to 5.

Solution



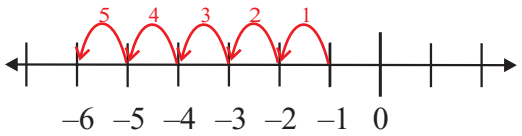
Counting **6 spaces to the left of 5** on the number line shows that the answer is **–1**.

When you're adding a negative number, it doesn't matter whether you're adding it to a **positive** or **negative** number — you always move to the **left**.

Example 3

Using the number line, add –5 to –1.

Solution



Counting **5 spaces to the left of –1** on the number line shows that the answer is **–6**.

Guided Practice

Answer the following Exercises without using a calculator.
Use a number line to help.

1. 7 + 3
2. –2 + 2
3. –8 + 4
4. 1 + 3
5. –81 + 8
6. –9 + 11
7. 9 + (–2)
8. 35 + (–10)
9. –5 + (–1)
10. 0 + (–5)
11. 36 + (–10)
12. –15 + (–7)

Subtracting a Positive is Like Adding a Negative

You already know that to add a negative integer, you **move that number of positions to the left** on the number line.

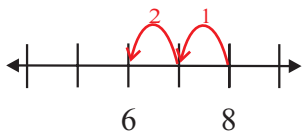
The method for **subtracting** a **positive** integer is exactly the same.

Example 4

Using the number line, calculate 8 – 2.

Solution

The 2 tells you that you need to move **2 steps**, and the "–" sign tells you that you need to move **left**. So the answer is **6**.



Guided Practice

Answer the following Exercises without using a calculator.
Use a number line to help.

13. 7 – 3
14. –1 – 10
15. 8 – 8
16. 3 – 8
17. –2 – 9
18. 1 – 10

Subtracting a Negative Is Like Adding a Positive

You've already seen that adding a negative means you move left on the number line. So **subtracting** a negative must mean you move **right** on the number line.

Another way of saying this is that **subtracting a negative** is the same as **adding a positive**.

Example 5

Calculate 8 – (–3).

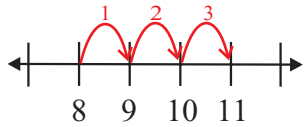
Solution

$$8 - (-3)$$

The – sign means "go left on the number line"...

...but this – sign reverses the direction again

You end up moving 3 steps right on the number line. So you can just write 8 – (–3) as 8 + 3 = **11**.



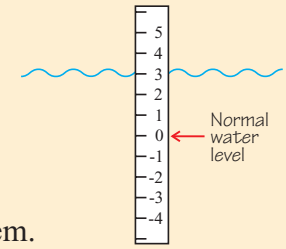
Guided Practice

Write Exercises 19–24 as additions, then find the solutions.

19. 14 – (–9)
20. 23 – (–10)
21. –9 – (–15)
22. 4 – (–4)
23. –4 – (–8)
24. –26 – (–17)

Independent Practice

1. The water level of a lake in early April was 3 inches above normal. The water level had dropped to –4 inches below normal by early May. What integer can be added to 3 to get the water level, in inches, in early May?



Write each of Exercises 2–5 as an addition problem.

2. 4 – 3
3. 7 – 8
4. –1 – 26
5. 4 – 5

Write each of Exercises 6–9 as a subtraction problem.

6. 5 + (–9)
7. 1 + (–3)
8. 2 + (–1)
9. 433 + (–348)

Find the answer to each of Exercises 10–13.

10. 4 + (–15)
11. 18 – 9
12. 23 + 6
13. 15 – (–3)

Now try these:
Lesson 1.1.2 additional questions — p428

Round Up

Adding to and subtracting from integers is a matter of deciding which direction to move, then moving the correct distance. When you're deciding whether to go left or right you only need to think about whether you're adding a positive or negative number — it doesn't matter what you're adding it to.

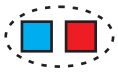
Section 1.2 introduction — an exploration into:

Multiplication with Integer Tiles

You can use colored tiles to show what happens when you multiply numbers together. Multiplication is really just the same as doing lots of addition — for example, 3 × 2 is the same as 2 + 2 + 2, or even 3 + 3.

In this Exploration, **blue tiles** show **positive numbers**, while **red tiles** show **negative numbers**. For example, the number **5** is represented by **5 blue tiles**, and **–4** is represented by **4 red tiles**.

A blue and a red tile together have a total **value of zero**, and are called a **zero pair**.



1 + (–1) = 0

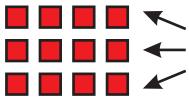
You can **multiply** numbers by adding together equal groups of tiles.

Example

Find 3 × (–4) using integer tiles.

Solution

This multiplication problem means: "**Make 3 groups of –4.**"
So **3 × (–4) = –12**.



3 groups of –4 (12 red tiles in total)

Making **3 groups** of something is easy to think about — it involves **adding** things together. Making **–4 groups** of something is harder to understand, but it involves **subtracting**.

Example

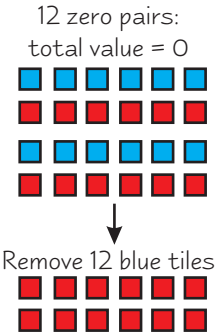
Find –4 × 3 using integer tiles.

Solution

This multiplication problem means: "**Make –4 groups of 3.**"

Making **–4 groups** of 3 is the same as **subtracting 4 groups** of 3 blue tiles. But before you can do this, you need some blue tiles. Use **zero pairs**.

Start with 12 zero pairs. Now **remove 4 groups** of 3 blue tiles, or 12 blue tiles. That leaves 12 red tiles, or –12. So **–4 × 3 = –12**.



Exercises

1. Use integer tiles to model and solve each of the following:

 a. 5 × 3 **b.** 2 × (–6) **c.** 4 × (–3) **d.** –4 × 2 **e.** –3 × (–2) **f.** –2 × (–5)

2. Use your results to find a rule for the product of a positive and a negative number.

3. Use your results to find a rule for the product of two negative numbers.

Round Up

Multiplying using red and blue tiles can help you work out the answers to some difficult questions. In this Section, you'll see other ways of thinking about multiplication with negative numbers. But the answers will always be the same as you get with tiles.

Lesson 1.2.1

Section 1.2
Multiplying with Integers

You've seen how to use a number line to show what happens when you add or subtract positive and negative integers. In this Lesson you'll see how it can be useful for doing multiplication problems too.

California Standard:
Number Sense 2.3
Solve addition, subtraction, **multiplication**, and division **problems, including those arising in concrete situations, that use positive and negative integers** and combinations of these operations.

What it means for you:
You'll see what happens when you multiply positive and negative whole numbers.

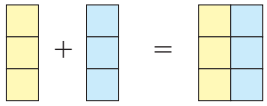
Key words:
- integer
- product
- factor

Don't forget:
You can work out the product of two numbers in either order — $a \times b = b \times a$.
This is called the commutative property of multiplication — see Lesson 2.3.2 for more.

Multiplication Is All About Grouping Things

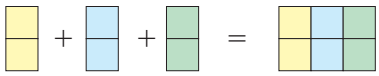
Multiplication is really a way of **adding together groups** of objects.

For instance, 2×3 just means **"2 groups of 3."**



There are 6 blocks in total, so $2 \times 3 = 6$.

Doing **"3 groups of 2"** gives the same result.



There are still 6 blocks, so $3 \times 2 = 6$.

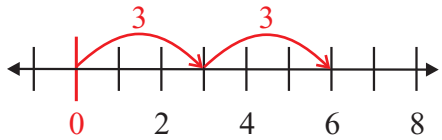
You can do the same kind of grouping and counting on the **number line**.

Example 1

Show the answer to 2×3 using a number line.

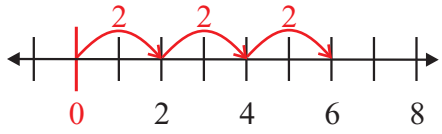
Solution

You can show the answer with **2 arrows**, each of **length 3**:



2 × 3 is twice as far from 0 as 3 is

You could also show the same answer with **3 arrows** of **length 2**:

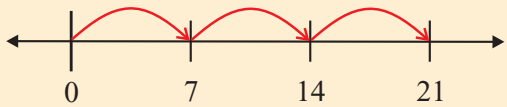


3 × 2 is three times as far from 0 as 2 is

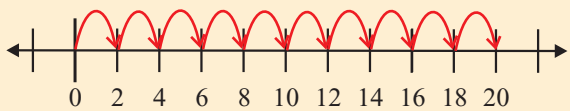
Guided Practice

What multiplication is shown on each number line?

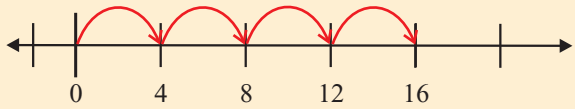
1.



2.



3.



Don't forget:
The answer to a multiplication is called a product.
The numbers being multiplied together are called factors.

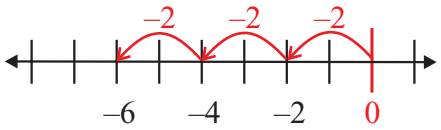
Multiplying by a Negative Changes the Direction

Even if you're multiplying by a negative, you're still dealing with **groups**.

So $3 \times (-2)$ still means **"3 groups of –2."**

Just like in Example 1, there are **3 arrows** of **length 2** on the number line, but this time the **negative** sign means they're pointing **left**.

Don't forget:
On the number line, positive numbers are on the right, negative numbers are on the left.

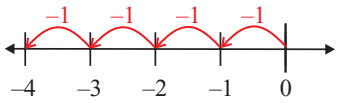


You can see from this number line that $3 \times (-2) = -6$.

Example 2

Calculate $4 \times (-1)$.

Solution



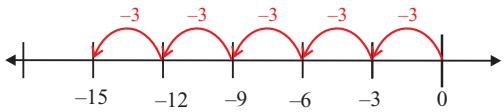
You can see from this number line that $4 \times (-1) = -4$.

Example 3

The outside temperature at midnight was 0 °F. Every hour after that, the temperature dropped by 3 °F. What was the temperature at 5 a.m.?

Solution

The **change** in temperature is **–3 °F** each hour for five hours, so you need to solve **5 × (–3)**.

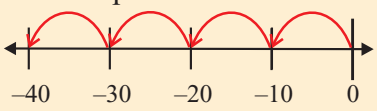


This shows that $5 \times (-3) = -15$, so at 5 a.m. the temperature was **–15 °F**.

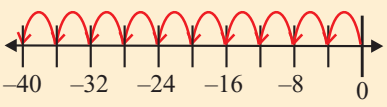
Guided Practice

What multiplication is shown on each number line?

4.



5.



6. What conclusion can you make from Exercises 4 and 5?

Calculate the following multiplications:

7. $7 \times (-4)$ **8.** -7×3 **9.** $30 \times (-2)$ **10.** -2×30

11. A submarine changes its depth in the water by –25 feet per minute. What is its total change in depth in four minutes?

12. A bird is flying toward the ground. Its height changes by –16 feet per second. What is the bird's total change in height in 5 seconds?

13. The amount of fuel in a racing car changes by –6 gallons per lap. What is the change in its fuel load over 7 laps?

Check it out:
A negative change such as in Exercises 11–13 usually shows something's getting smaller or lower.

A Negative Times a Negative Equals a Positive

On the last page you saw that multiplying a positive integer by a negative integer results in a negative solution. But if you multiply one **negative number** by **another**, their **"–" signs cancel** each other out.

Example 4

Calculate $-3 \times (-2)$.

Solution

You know that $3 \times (-2)$ means "3 groups of –2", and $3 \times (-2) = -6$.

The extra negative sign in $-3 \times (-2)$ just changes the sign again.
The answer must be positive: $-3 \times -2 = \mathbf{6}$

If you've got several negative integers to multiply, you can do it **bit by bit**.

Example 5

Calculate $-3 \times (-2) \times (-5)$.

Solution

$[-3 \times (-2)] \times -5$ **Work it out in smaller parts**

$= 6 \times (-5)$ **First multiply two of the numbers: –3 × (–2) = 6**

$= -30$ **Now, positive × negative = negative**

So you can multiply **any** two integers using these rules:

Rules for multiplying integers

positive × positive = positive
positive × negative = negative
negative × positive = negative
negative × negative = positive

You can still use these rules even if you're multiplying more than two numbers together. Just **count the number of "–" signs** in the question.

If there's an **even** number of negative factors, they'll cancel out in pairs, and the answer will be **positive**.

If there's an **odd** number of negative factors, you'll end up with one that doesn't cancel out, so the final answer will be **negative**.

Example 6

Solve $-2 \times 5 \times (-4) \times (-10)$.

Solution

There are **three minus signs**. This is an odd number, so the answer will be **negative**. Work out the "size" of the number by finding:

$$2 \times 5 \times 4 \times 10 = 400$$

So the answer must be **–400**.
To prove it, break the question down into smaller parts:

$-2 \times 5 \times (-4) \times (-10)$

$= -10 \times (-4) \times -10$ **Negative × positive = negative**

$= 40 \times (-10)$ **Negative × negative = positive**

$= -400$ **Positive × negative = negative**

Guided Practice

Say whether the following will give positive or negative answers.
(You don't need to work out the actual solutions.)

14. $-8 \times (-3)$
15. -2×9
16. $2 \times (-3) \times (-5)$
17. $-27 \times (-13) \times (-7) \times (-17)$
18. $-6 \times 11 \times (-19) \times (-83)$
19. $-1 \times 2 \times (-3) \times 4 \times (-5)$
20. $225 \times (-311) \times (-277) \times (-1008) \times 47 \times (-119)$

Independent Practice

In Exercises 1–6, use a number line to solve the multiplication.

1. 5×7
2. -3×12
3. 11×4
4. $6 \times (-6)$
5. $21 \times (-2)$
6. $-8 \times (-3)$

7. Ms. Ross is overdrawn on her bank account. Her balance is –$30. Mr. Banks is overdrawn on his bank account by 5 times the amount Ms. Ross is overdrawn. What is Mr. Banks's account balance?

8. Sara multiplied two negative integers together. She then multiplied her answer by another negative number. Is her final result positive or negative?

9. Pablo multiplied two integers together. The answer that he got was –28. What integers might he have multiplied together?

Now try these:
Lesson 1.2.1 additional questions — p428

Round Up

It's important to know what happens when you multiply by negative integers, because they appear in lots of math topics. You'll need the rules for multiplying again when you learn about dividing with negative integers in the next Lesson.

Lesson 1.2.2

Dividing with Integers

California Standard:
Number Sense 2.3
Solve addition, subtraction, multiplication, and **division problems, including those arising in concrete situations, that use positive and negative integers** and combinations of these operations.

What it means for you:
You'll learn what to do when you see positive and negative integers in division questions.

Key words:
- integer
- quotient
- positive
- negative

You already know how to divide by positive whole numbers. Now you're going to see how a number line can be helpful in answering division questions with positive and negative integers. This is similar to what you saw in the previous Lesson with multiplication.

Division Means Breaking a Number into Equal Parts

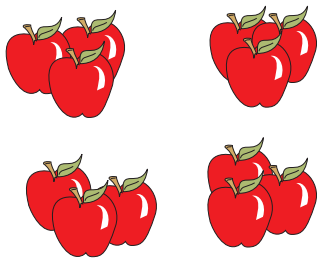
"**Divide**" is another way of saying "**share out equally**."

Example 1

A large bag contains 12 apples. The apples are shared equally among 4 students. How many apples does each student get?

Solution

You can see from the picture that if the apples are **shared out equally**, each student will get **3 apples**.



You can also think of division as the **opposite** of **multiplication**.

Example 1 is asking you to find **12 ÷ 4**.

You can rewrite that as $12 \div 4 = ?$
which is the same as saying $4 \times ? = 12$

Then you can use **"guess and check"** to work out what the question mark must represent:

Try substituting 1: $4 \times 1 = 12$ ×
Try substituting 2: $4 \times 2 = 12$ ×
Try substituting 3: $4 \times 3 = 12$ ✓

Check it out:
When you do a division, the result is called the quotient.

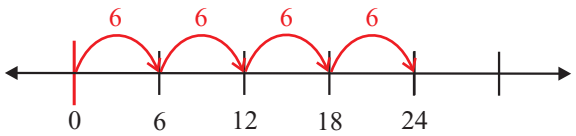
Guided Practice

Solve without a calculator:

1. $18 \div 3$ **2.** $21 \div 7$ **3.** $12 \div 6$
4. $26 \div 2$ **5.** $100 \div 4$ **6.** $54 \div 9$
7. $88 \div 11$ **8.** $70 \div 5$ **9.** $48 \div 12$

Division Can Also Be Shown on a Number Line

Below is a number line showing the multiplication $4 \times 6 = 24$.



You can think about this line in another way. It shows that the number 24 can be **divided** into 4 **equal parts**, each of size 6. It also shows that the number 6 is **one-fourth** of the way from 0 to 24.

So the number line also shows that $24 \div 4 = 6$

Check it out:

One-half of a number means dividing it by 2.

One-third of a number means dividing it by 3.

One-fourth of a number means dividing it by 4.

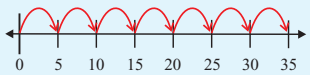
And so on...

Check it out:

There are two ways of using a number line to represent a division like $35 \div 5$.

1) You can do it as in Example 2, by saying "$35 \div 5$ means you have to divide 35 into 5 equal parts."

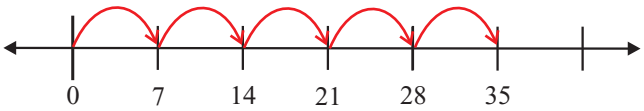
2) Or you can say "$35 \div 5$ means working out how many 5s there are in 35."
This gives you a slightly different number line, as shown below.



Both ways are equally good, but in this book, we've used approach 1) above.

Example 2

Use the number line below to solve $35 \div 5$.

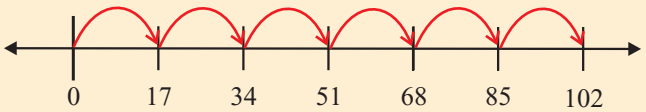


Solution

The number line shows the line from 0 to 35 divided into **5 equal parts**. You can see that **7** is **one-fifth** of the way from 0 to 35, so the answer must be $35 \div 5 = \mathbf{7}$.

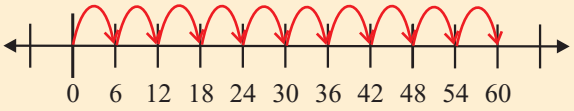
Guided Practice

Use the number line below to answer Exercises 10–12:



10. $102 \div 2$ **11.** $102 \div 3$ **12.** $85 \div 5$

Use the number line below to answer Exercises 13–18:



13. $24 \div 4$ **14.** $60 \div 5$ **15.** $54 \div 3$
16. $36 \div 2$ **17.** $36 \div 3$ **18.** $48 \div 4$

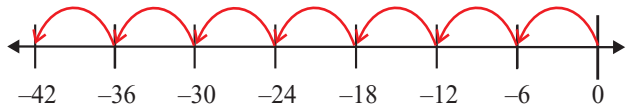
Negative Numbers Can Be Used in Division

Dividing with **negative** integers is **similar** to dividing with **positive** integers. You still have to work out how many **equal parts** go into the number.

Example 3

Calculate –42 ÷ 7 using the number line.

Solution



The number **–6** is **one-seventh** of the way from 0 to –42 on the number line. So the answer is **–6**.

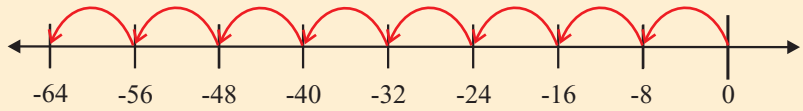
You can check this by working out 7 × (**–6**) = –42

Check it out:

Another way of thinking about Example 3 is to say that –42 can be split into 7 equal groups of –6.

Guided Practice

Use the number line below to answer Exercises 19–22:



19. –64 ÷ 8

20. –64 ÷ 2

21. –40 ÷ 5

22. –48 ÷ 3

The **rules** for dividing integers are really similar to the **multiplication** rules you saw in the last Lesson.

Rules for dividing integers

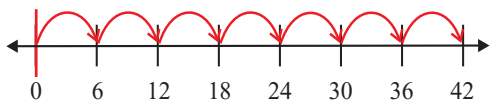
positive ÷ positive = positive
positive ÷ negative = negative
negative ÷ positive = negative
negative ÷ negative = positive

Example 4

Calculate –42 ÷ (–7).

Solution

You can use a number line to work out that 42 divides into 7 equal groups of 6, so 42 ÷ 7 = 6.



Then using the above rules, you can see that –42 ÷ (–7) = **6**

Check it out:

Another way of thinking about Example 4 is to say that –42 ÷ (–7) means "how many equal groups of –7 are in –42?".

Example 5

Calculate –120 ÷ (–40).

Solution

You can use the **rules for dividing integers** instead of the number line. You are dividing a **negative by a negative**, so the answer is **positive**. The answer will be the same as 120 ÷ 40.

$$-120 \div (-40) = 120 \div 40 = 3$$

Check: $-40 \times 3 = -120$

Guided Practice

Say whether the following will give positive or negative answers. (You don't need to work out the actual solutions.)

23. –18 ÷ 3 **24.** –27 ÷ (–9) **25.** –17 ÷ (–1)

26. 625 ÷ (–25) **27.** –363 ÷ 11 **28.** –1008 ÷ (–24)

Independent Practice

Now try these:
Lesson 1.2.2 additional questions — p429

Solve:

1. 16 ÷ 2 **2.** 45 ÷ 5 **3.** –36 ÷ 12

4. 66 ÷ (–11) **5.** –27 ÷ (–9) **6.** –126 ÷ 9

Use a number line to solve:

7. 63 ÷ 7 **8.** 371 ÷ 7 **9.** 40 ÷ (–5)

10. –126 ÷ 6 **11.** 188 ÷ (–4) **12.** –64 ÷ (–4)

13. The Garcias are going to the store. Mrs. Garcia is going to divide $42 equally among her three children so they have pocket money. How much money does each child get?

14. A submarine is –400 m from the surface of the ocean. It rises to half of this depth. At what depth is the submarine now?

15. Emma's bank account is overdrawn. Her balance is –$240. She pays some money into the account so that her overdraft is reduced to one-third of the old balance. What is the new balance on Emma's account?

Round Up

You've now learned rules for adding, subtracting, multiplying, and dividing integers.
In the next Lesson, you'll see these rules applied in real-life situations.

Lesson 1.2.3

Integers in Real Life

California Standard:
Number Sense 2.3
Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations, that use positive and negative integers and combinations of these operations.

What it means for you:
You'll answer real-life questions that involve addition, subtraction, multiplication, and division.

Key words:
- integer
- range
- operation

Check it out:
Remember to write the solution as an answer to the original question — for instance, in Example 1 you must remember to add the °F symbol.

Check it out:
To find the range of a set of numbers, you subtract the least from the greatest.

Now you've learned about positive and negative integers, and how to add, subtract, multiply, and divide them. In this Lesson you'll use what you have learned in real-life problems.

Real Situations Often Involve Addition and Subtraction

You always need to **read** the problem and **identify** the operations being used — either add, subtract, multiply, or divide. Then you should always **rewrite** the question in **math language** to help you solve it.

Example 1

At 10 a.m. the outside temperature was 70 °F. It was 5 °F warmer by midday. What was the midday temperature?

Solution

The question involves **adding** 5 °F to 70 °F, which is written **70 + 5**.

So the midday temperature was 70 + 5 = **75 °F**.

Integer subtraction can be used to find out how much a quantity has **changed**. It can also be used to calculate the **range** of a set of numbers.

Example 2

A student in Ms. Chang's English class scored 12 points on a quiz. Another scored 27 points. What is the range of those scores?

Solution

The range is found by subtracting 12 from 27.

So the answer is 27 – 12 = **15**.

Guided Practice

1. Jerry has 23 ballpoint pens. His friend George gives him 16 more pens. How many pens does Jerry have now?

2. One day there are 32 passengers on a bus. The next day there are 67. What is the range of these two numbers?

3. A scuba diver is 16 meters below sea level. How far below sea level is the diver if she goes 15 meters lower?

4. Althea has 37 bags of potato chips. She gives 10 to Jackie. How many bags does Althea have now?

5. In one day, Isaac runs the 400 m and 1600 m races. How many meters has he run during these two races?

6. After a drought, the water level of Lake Pinebrook was 19 inches below normal. After a long rainy season, the water level rose to 22 inches above normal. Find the range of the water levels.

Watch Out for Multiplication and Division Problems

Look out for words like "times" or "double" or "triple" — these words are **clues** that the problem will involve **multiplication**.

Example 3

A small fish is swimming 5 feet below the surface of the ocean. A larger fish is swimming at 12 times the depth of the small fish. How far below the surface is the larger fish?

Solution

The question tells you how deep the small fish is. You need to find **12 times** that figure, so you need to **multiply**.

Written as a multiplication problem, the question is $5 \times 12 = \mathbf{60}$.

The larger fish is **60 feet** below the surface of the ocean.

Division problems often have the word **"divided"** in them, or they may have a **fraction** written out in words.

Example 4

A large fish is swimming 145 feet below the surface of the ocean. A smaller fish is swimming at one-fifth the depth of the large fish. How far below the surface is the smaller fish?

Solution

This time you need to find **one-fifth** of 145, so you need to **divide**.

Written as a division problem, the question is $145 \div 5 = \mathbf{29}$.

The smaller fish is **29 feet** below the surface of the ocean.

Guided Practice

7. Kia tripled her CD collection this year. She started with 36 CDs. How many does she have now?

8. It takes Marcus 2.5 hours to paint his bathroom, but 5 times longer to paint his hall. How long does he spend painting the hall?

9. A hot-air balloon is 96 feet from the ground. It then sinks to one-quarter of that height. What is the new height of the balloon?

Some Questions Involve More Than One Operation

Sometimes you'll need to carry out **more than one operation** to find the answer to a question. If that happens, you will need to break the question up into parts, then solve them in the right order.

Check it out:

Watch out for clue words that will help you figure out what to do to solve the question. Here are some examples of clue words:

Add – sum, total, increase, more than

Subtract – take away, fewer, difference, decrease, less than

Multiply – times, product, double, triple

Divide – share, quotient, split, one-half, one-third

Example 5

A bird is flying 25 feet above the ground. The bird triples its height, and then drops 60 feet to meet another bird. How high above the ground are the birds when they meet?

Solution

This question has **two** parts.

"The bird **triples** its height" means you need to **multiply by 3**:

$$25 \times 3 = 75 \text{ feet.}$$

A **drop** of 60 feet means **subtract** 60:

$$75 - 60 = \mathbf{15}$$

The birds are **15 feet** above the ground when they meet.

Guided Practice

For Exercises 10–13, decide what operations are needed, then write out the question as a math problem and solve it.

10. Rory is driving his car at a speed of 40 mph. Before getting on the highway, he reduces his speed by one-half, then triples his speed when he gets on the highway. What is Rory's speed on the highway?

11. Kelly has $12. Her friend Elena lends her another $28. Kelly then spends one-fourth of her money on a book. How much does the book cost?

12. After low sales in January, a clothing store was overdrawn and had a bank balance of –$2400. In February, the store was overdrawn by one-third of what it was in January. In March, the overdraft was one-fourth of what it was in February. What was the balance in March?

13. A fire truck holds 500 gallons of water. A hose attached to the truck uses 12 gallons of water each minute to put out a fire. The hose was turned on for 8 minutes to put out a fire. How much water is left in the fire truck after the fire?

Independent Practice

Now try these:
Lesson 1.2.3 additional questions — p429

1. Jermaine is baking. He makes an extra-large pie by multiplying the amounts of ingredients in the recipe by three. If the original recipe says to use 350 grams of apples, how many grams of apples does Jermaine use?

2. A bank clerk notices two accounts that are overdrawn. Account 1 has a balance of –$24, and account 2 is overdrawn 3 times that amount. What is the balance of account 2?

3. On Monday, Laura rides 21 miles on her bike. On Tuesday, she rides one-third the distance she did on Monday. On Wednesday, she rides for 10 miles more than on Tuesday. How far did Laura ride on Wednesday?

4. Three boats set out from a harbor. Boat A travels 112 miles. Boat B travels 45 miles less than boat A. Boat C travels 5 times as far as boat B. How far did boat C travel?

5. Karl has 99 red balloons. He gives one-third to Libby, then another 20 each to Billy and Rosario. How many balloons does Karl have left?

6. Manuel plants a 4-foot-tall tree in his backyard. Ten years later, the tree is five times as tall as when he planted it. The tree is then struck by lightning, and the top half breaks off exactly halfway up the trunk. How tall is the tree after it is struck by lightning?

7. Susan, Stephanie, Connor, and Gino open a store together. The business is not successful and makes a loss of $19,500. After Paul buys their stock for $1500, the rest of the loss is split between the partners. How much money has each partner lost?

8. The water level in a lake drops during the summer to 23 inches below its normal level. During a period of heavy rain, the water level rises again by 5 inches per day. After 7 days of rain, how far above the normal level is the water in the lake?

9. Ms. Alano has a box of 100 paper clips. She gives 3 to each of her students, and then has 10 left over. How many students does Ms. Alano have?

10. Doug and Jim are playing golf. They both start off 300 yards away from the hole. Jim's ball lands 175 yards from the hole. Doug's ball lands half as far away as Jim's, but then rolls down a slope which takes it 30 yards farther away from the hole. How far from the hole does Doug's ball finish up?

Round Up

Remember to read the question carefully, think about what you're being asked, and convert everything into math language to help you solve it. The more you practice, the better you'll get at spotting the different clue words for addition, subtraction, multiplication, and division.

Section 1.3 introduction — an exploration into:

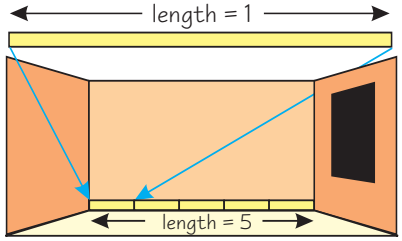
A Decimal Strip

In this Exploration, you're going to measure things in meters. You can measure any length in meters. Big things (like a classroom) might be many meters long. But small things (like your pencil) probably measure much less than one meter — this is where decimals are useful.

Look at your **meter strip** — its length in meters is **1**.

To measure the length of the **classroom**, you could lay several strips end to end. The length of the classroom would be the number of strips you use.

But you need a different method for things **less than a meter long**.



Example

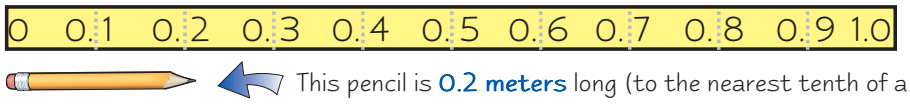
Show how you could measure the length of your pencil using your meter strip.

Solution

You could divide your strip into **smaller** sections.

If you divide the strip into **ten equal sections**, each section will be **one tenth**, or **0.1**, of a meter. Fold your strip into ten equal parts. Label each of the folds using tenths: **0.1, 0.2, 0.3, ..., 1.0**

Now you can use your meter strip to measure things to the nearest **tenth of a meter**.



Exercises

1. Using your strip, measure five objects in the classroom to the nearest tenth of a meter.

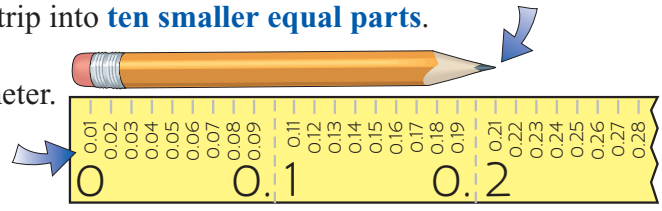
Example

Show how you could make your strip even more accurate for measuring.

Solution

You can divide each small section on your strip into **ten smaller equal parts**.
This means the whole strip is in **100 parts**.
Each part is **one hundredth**, or **0.01**, of a meter.

Divide your strip like this one, using a ruler.
Label each mark as shown.



Now you can use your meter strip to measure things to the nearest **hundredth of a meter**.

Exercises

2. Measure the same five objects as in Exercise 1, but to the nearest hundredth of a meter.

Round Up

Each hundredth of your strip was one centimeter long. You could divide each centimeter into ten smaller parts. Then each part would be 0.001 (one thousandth) of a meter long (a millimeter).

Lesson 1.3.1

Section 1.3 Decimals

You've seen decimals in earlier grades — but now you're going to deal with different place values and negative decimals too.

California Standard:

Number Sense 1.1

Compare and order positive and negative fractions, **decimals**, and mixed numbers **and place them on a number line**.

What it means for you:

You'll learn more about the place values of decimals.

Key words:

- decimal point
- tenth
- hundredth
- thousandth

Check it out:

To read a decimal number, say the whole-number part in the normal way — and then add the part after the decimal point, reading it as the number of tenths, or the number of hundredths, or similar. Also, say the word "and" to make it clear where the decimal place is.

For example, 5.7 is read "five and seven tenths," while 12.75 is read "twelve and seventy-five hundredths."

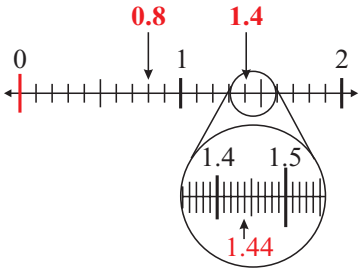
Decimals Are Numbers Between Integers

Decimals are used to represent numbers between integers. Digits after the decimal point represent part of an integer.

Each **one** on the number line is divided into **10 tenths**.

0.8 means "**0 plus 8 tenths,**" and 1.4 means "**1 plus 4 tenths**."

Each **tenth** is then divided into **10 hundredths**. In other words, each **one** is divided into **100 hundredths**.



Similarly, each **one** on the number line is divided into **1000 thousandths**.

Example 1

Explain the meaning of each of the digits in the number 6.582. Between which two integers does the number 6.582 lie?

Solution

- The number 6.582 means: **6 ones** plus **5 tenths** plus **8 hundredths** plus **2 thousandths**.
- The number 6.582 is **6 ones** plus **numbers after a decimal point**. So the number 6.582 lies **between 6 and 7**.

Example 2

David spent 1 hour and 30 minutes playing basketball. Express this number of hours as a decimal.

Solution

- 30 minutes is 0.5 of an hour, so David played basketball for 1 whole hour plus an extra 0.5 hours.
- Putting those together, David played basketball for **1.5 hours**.

Guided Practice

In Exercises 1–4, first say which digit is in the hundredths place, then which digit is in the thousandths place.

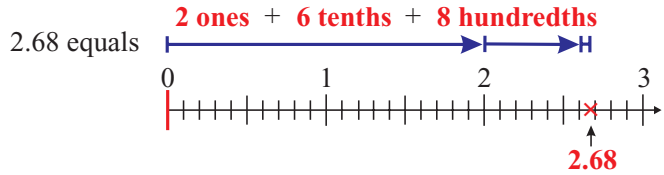
1. 34.251 **2.** 128.734

3. 0.163 **4.** 3514.902

5. A runner completed a marathon in 4 hours and 15 minutes. Express this number as a decimal.

Decimals Can Be Shown on the Number Line

To put a decimal on the number line, **read it as an instruction**.
For example, with the number **2.68**:
- first move **2 ones** along the number line,
- then move **6 tenths** in the same direction,
- then move **8 hundredths** in the same direction.



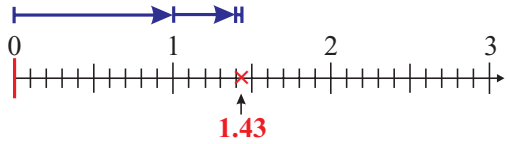
Check it out:
The numbers 1.230 and 1.23 are the same because the 0 on the end of 1.230 just means "1.23 and then 0 thousandths extra."
For the same reason, any number of 0s can be added or removed if they come at the very end of a number after a decimal point.

Example 3

Put 1.43 on the number line.

Solution

1.43 is 1 along the number line, then 4 tenths, and then 3 hundredths.



Check it out:
If the number line is only divided into tenths, you'll have to estimate the hundredths position by placing the cross between the two lines.

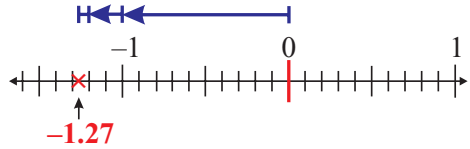
You can plot **negative** decimals in the same way.
But this time you always move in the **negative direction** — to the **left**.

Example 4

Put –1.27 on the number line.

Solution

–1.27 is 1 along the number line **in the negative direction**,
and then 2 tenths in the same direction,
and then 7 hundredths in the same direction.



Guided Practice

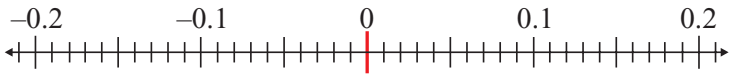
Place the numbers in Exercises 6–9 onto a number line.

6. 1.4
7. –2.8
8. 6.3
9. –0.5

Draw Zoomed-in Number Lines to Be More Accurate

It is hard to show thousandths on a normal number line **accurately**. Instead, you can draw a **zoomed in** number line.

This one has a scale marked in **tenths** rather than integers, with divisions for **hundredths**.

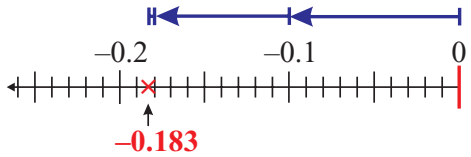


Example 5

Put –0.183 on the number line.

Solution

–0.183 is 1 tenth along the number line **in the negative direction**, and then 8 hundredths in the same direction, and then 3 thousandths in the same direction.



Check it out:

Example 5 uses a "close up" number line that only shows the origin and a small section of the number line immediately to the left. It's as though someone has zoomed in on the number line, allowing you to see the really small hundredth marks.

Guided Practice

Place the numbers in Exercises 10–13 onto a number line.

10. –1.73 **11.** –5.78

12. –5.624 **13.** 0.597

Independent Practice

1. The temperature in Celsius of a substance during a science experiment is twelve and twenty-three hundredths below zero. Represent this temperature using a decimal.

Place the numbers in Exercises 2–5 onto a number line.

2. 23.5 **3.** –2.28

4. –19.23 **5.** 3.246

Now try these:

Lesson 1.3.1 additional questions — p429

Round Up

Decimals look complicated, but they can be broken up into tenths, hundredths, and thousandths, and put on the number line. In the next Lesson you'll see how that helps you put decimals in order.

Lesson 1.3.2

Ordering Decimals

California Standard:

Number Sense 1.1

Compare and order positive and negative fractions, **decimals**, and mixed numbers **and place them on a number line.**

What it means for you:

You'll learn two different methods for ordering decimals.

Key words:

- greater than (>)
- less than (<)

Don't forget:

You saw the "<" and ">" symbols in Lesson 1.1.1, when you were comparing integers.

Now it's time to learn how to order decimals. If you're given two decimals, you need to be able to say which is greater.

There are a couple of ways of ordering decimals — either using the trusty number line or without it.

The Number Line Shows the Order of Decimals

In Lesson 1.1.1 you saw that if any two integers are plotted on the number line, **the number to the left is less than the number to the right**. The same thing applies to **decimals** — and it doesn't make any difference whether the numbers are positive or negative.

The symbol **<** is used to mean "**less than**" and the symbol **>** is used to mean "**greater than**."

Example 1

Using the number line, explain why the statement $-1.7 > 1.2$ is **not true**.

Solution

Place –1.7 and 1.2 on a number line:



–1.7 is **to the left** of 1.2, so **–1.7 is less than 1.2**.

So $-1.7 < 1.2$, which means it is **false** to say that $-1.7 > 1.2$.

Guided Practice

In Exercises 1–4, use a number line to say which number is greater.

1. 23.45, 10.2
2. –2.3, 1.9
3. 8.163, 8.162
4. –1.2, –0.8

Use a number line to say whether each statement in Exercises 5–8 is true or false.

5. $3.9 < 3.14$
6. $2.1 > 2.09$
7. $0.5 < -1.5$
8. $-1.3 > -1.2$

Positive Decimals Can Be Compared Digit by Digit

You can also compare decimals **without** using the number line.

Don't forget:
Adding 0s to the end of decimals doesn't change the value at all. It just means "...and no extra tenths, hundredths, or thousandths."

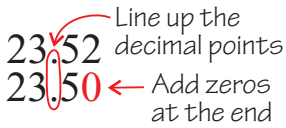
Example 2

Without using the number line, say whether 23.52 is greater than 23.5.

Solution

Before you compare, make sure the **decimal points line up**.

Add zeros to the end of either decimal to make them the same length.





Work from left to right and **compare each of the digits**. The first three digits are the same in both numbers.

The digits in the hundredths column are different, so use them to find the answer. **2 is greater than 0**, so **23.52 is greater than 23.5**.

Guided Practice

In Exercises 9–12, say which is the lesser number.

9. 0.23, 0.24
10. 3.5, 13.5
11. 2.5, 2.23
12. 0.1, 0.01

13. By comparing each with the others, put these decimals in order, from the least to the greatest:

23.45, 0.356, 3829.34, 7.5, 0.2

Negative Decimals Can Be Compared Too

Comparing **negative decimals** is the same as comparing positive decimals.

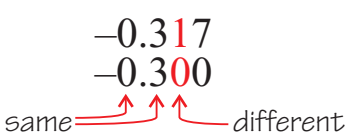
Don't forget:
0 isn't positive or negative.

Example 3

Without using the number line, say which of –0.3 and –0.317 is greater.

Solution

Add zeros to –0.3 to make it the same number of decimal places as –0.317, then compare each digit starting from the left.



The first two digits are the same in both numbers. The digits in the hundredths column are different, so they can be used to find the answer. **But remember that these digits are all negative.**
0 is greater than –1, so **–0.3 is greater than –0.317**.

Don't forget:

Negative numbers always go on the left of the origin, and positive numbers always go on the right of the origin. So, positive numbers are always greater than negative numbers, whether they are decimals or not.

Guided Practice

In Exercises 14–17, say which is the greater number.

14. –0.003, –0.017 **15.** –2.1, 13.5

16. –18.9, –25.73 **17.** –0.01, –0.001

18. By comparing each with the others, put these decimals in order, from the least to the greatest:

–2.0, –12.7, –194.644, –0.5, –7.0

Independent Practice

1. Explain why you can say that –3.234 is less than 2.855 without using a number line or comparing any digits.

Say whether each statement in Exercises 2–7 is true or false.

2. $23.3 > 23.25$ **3.** $4.83 > 4.9$

4. $-34.234 > -2.334$ **5.** $-4.234 < 0.001$

6. $14.2 < 19.3$ **7.** $15.345 < 393.3$

8. By comparing each with the others, put these decimals in order:

–3.344, 28.0, –0.7, 0.523, 0.001, 18.534, –3.345, 0.01

9. Adan works for a delivery company. A customer comes in with a package that weighs 1.023 pounds. Adan knows that packages that weigh less than 1.7 pounds cost $2 but packages that weigh 1.7 pounds or more cost $3. How much should he charge the customer?

10. Helen and Morty go scuba diving. Helen swims down 10.9 m below the surface of the water. Morty swims to 10.28 m. Who is lower?

11. Annie is teaching Alvy how to drive. He looks at the fuel gauge to check if they need to stop for gasoline, and sees this:



The gauge shows how much gasoline is left in gallons. How much is left?

Exercises 12–14 make use of the table below.

Mrs. Bueller is observing a substance as part of a biotech experiment at work. The table shows the temperatures of the substance on Monday, Tuesday, and Wednesday.

Day	Temperature
Monday	–20.3 °C
Tuesday	21.5 °C
Wednesday	19.8 °C

12. On what day was the temperature the coldest?

13. On what day was the temperature the warmest?

14. The temperature was 2 and 25 hundredths warmer on Thursday compared to Wednesday. What was the temperature of the substance on Thursday?

Now try these:

Lesson 1.3.2 additional questions — p430

Round Up

Comparing decimals can be tricky. Just make sure that before you compare numbers digit by digit, you make sure the decimal points are correctly lined up.

Lesson 1.4.1

Section 1.4
Rounding Numbers

Rounding involves replacing one number with another number that's easier to work with. You'll use rounded numbers in the next couple of Lessons to check and to estimate answers.

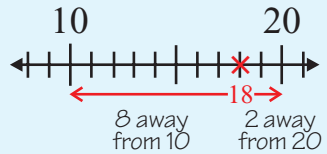
California Standard:
Mathematical Reasoning 2.1 Use estimation to verify the reasonableness of calculated results.

What it means for you:
You'll learn about rounding exact figures to make them easier to work with.

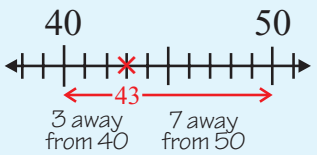
Key words:
- rounding
- place value
- digit
- decimal
- tenth
- hundredth

Don't forget:
A multiple of ten is any number that ends in 0: 10, 20, 100, 2500, 58340, and so on.

Check it out:
If you place 18 on the number line, you can see that it's nearer to 20 than 10.
So 20 is the nearest ten to 18.



43 is nearer to 40 than 50.
So 40 is the nearest ten to 43.



Rounded Numbers Can Be Easier To Use

Suppose you wanted to find 18 × 43, but had lost your calculator.
You could find an answer **close to** 18 × 43 by **rounding to the nearest ten**.

"**Rounding to the nearest ten**" means **replacing a number with the nearest multiple of 10**.
Replacing a number with a **higher** number is called **rounding up**.
Replacing a number with a **lower** number is called **rounding down**.

Example 1

Round 18 and 43 to the nearest ten.

Solution

You need to decide whether to round up or down.
Look at the digit in the **ones place**:
- If the ones digit is **5 or more**, round **up**.
- If the ones digit is **4 or less**, round **down**.

Start with 18: The digit in the ones place is **8**.
8 is **more than 5**, so **round up**.
18 rounded up to the nearest ten is **20**.

Next, 43: The digit in the ones place is **3**.
3 is **less than 4**, so **round down**.
43 rounded down to the nearest ten is **40**.

By rounding, you can replace **18 × 43** with **20 × 40**.
This is much easier to solve: 20 × 40 = **800**
800 is fairly close to the real answer: 18 × 43 = **774**

Guided Practice

In Exercises 1–8, round the numbers to the nearest ten.

1. 36	**2.** 84	**3.** 199	**4.** 4006
5. 267	**6.** 7161	**7.** 2994	**8.** 2995

9. During a science experiment, a group of students observed that there were 415 ants in a colony. Round this amount to the nearest ten.

You Can Round to Different Place Values

You can round numbers to place values other than tens.

Write the number. Underline the digit in the position you want to round to.
- If the digit to the right of the underlined digit is **5 or more**, round **up**.
- If the digit to the right of the underlined digit is **4 or less**, round **down**.

Example 2

Round 25,281 to the nearest hundred.

Solution

Write the number, and underline the **hundreds** digit: 25,281

You're rounding to the nearest **hundred**, so that's going to be either **25,200** or **25,300**.

The digit to the **right** of the underline is **8**.
That's **greater than 5**, so round **up**. 25,**300**

So 25,281 rounds up to **25,300**, to the nearest hundred.

Guided Practice

In Exercises 10–13, round the numbers to the nearest hundred.
10. 38,383 **11.** 5756 **12.** 8128 **13.** 40,079

In Exercises 14–17, round the numbers to the nearest thousand.
14. 11,905 **15.** 8117 **16.** 2,599,582 **17.** 464,333

18. Clara lives in a city that has a population of 82,458 people. Write this population rounded to the nearest thousand.

You Can Round Decimals Just Like Whole Numbers

You round **decimals** in the same way as whole numbers. Instead of rounding to the nearest **ten, hundred,** and so on, you round to the nearest **one, tenth, hundredth,** or any other number of decimal places.

Example 3

Round 0.0815 to the nearest thousandth.

Solution

Write the number, and underline the **thousandths** digit: 0.0815

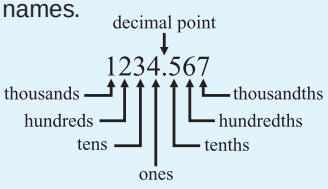
The nearest thousandth will be either **0.081** or **0.082**.

Look to the **right** of the underline.
The digit to the right is **5**, so round **up**. 0.082

So 0.0815 rounds up to **0.082**, to the nearest thousandth.

Don't forget:

The positions after the decimal point have special names.



See Lesson 1.3.1 for more about decimals.

Check it out:

If you're rounding decimals, don't add zeros to the right of the underlined digit. So 0.83 rounded to the nearest tenth is 0.8, not 0.80.

Guided Practice

Check it out:
Always read the question carefully. Don't mix up hundreds and hundredths, or thousands and thousandths.

In Exercises 19–22, round the numbers to the nearest tenth.

19. 28.0634 **20.** 2.247 **21.** 5.78 **22.** 6.892

In Exercises 23–26, round the numbers to the nearest hundredth.

23. 0.1066 **24.** 15.596 **25.** 409.4902 **26.** 7.734

In Exercises 27–30, round the numbers to the nearest thousandth.

27. 9.46071 **28.** 1.7254 **29.** 5.226822 **30.** 3.1007

31. A distance of 1 mile is equal to 1.609344 km. Write this to the nearest hundredth of a kilometer.

Exercises 32–34 are about Malik, who has $12.57 in his pocket.

32. How much money does Malik have to the nearest dollar?
33. How much money does Malik have to the nearest dime?
34. How much money does Malik have to the nearest quarter?

Independent Practice

Now try these:
Lesson 1.4.1 additional questions — p430

In Exercises 1–5, round the number 94,521.8375:

1. to the nearest hundred
2. to the nearest hundredth
3. to the nearest thousandth
4. to the nearest thousand
5. to the nearest one

6. The number 3478 was rounded to 3480.
To what place value was the number rounded?

7. Raul's thermometer shows that the temperature is 91.5 °F.
What is the temperature to the nearest degree?

8. Mount Whitney is 14,505 feet high.
Write this figure to the nearest hundred feet.

9. The average distance from the Earth to the Moon is 238,857 miles.
What is this distance to the nearest thousand miles?

10. The speed of light is 299,792,458 meters per second.
What is this to the nearest million m/s?

11. Jessica has $17.33. What is this amount to the nearest quarter?

12. A square inch is equal to 6.4516 cm^2. Convert 6 in^2 to cm^2, then round your answer to the nearest hundredth.

Round Up

Rounding numbers makes them easier to deal with. You can use rounded numbers to quickly find an approximate answer when you don't need an exact one. They can also help you to check your work, by letting you get an idea of how big your answer should be.

Lesson 1.4.2

Using Rounded Numbers

This Lesson will tell you more about using rounded numbers.
You'll think about how much certain numbers should be rounded.
You'll also see how rounded numbers are useful for checking your work.

California Standards:

Mathematical Reasoning 2.1
Use estimation to verify the reasonableness of calculated results.

Mathematical Reasoning 2.6
Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.

What it means for you:

You'll learn when it's a good idea to use rounding, and how to check your answers using rounded numbers.

Key words:

- rounding
- place value
- accuracy
- check
- reasonable
- estimate

Check it out:

The important thing when you choose how much to round is how much information you need to give.
If you don't round enough, the number might give details that aren't useful.
If you round too much, you might give information that is misleading.

Sometimes You Need to Choose How Much to Round

People **round** numbers to different **place values** depending on what the numbers are being **used for**.

Example 1

Below are three situations connected with the distance between Town A and Town B. Match each situation to the most suitable level of rounding.

Situation	Distance
A road sign in Town A shows the distance to Town B.	203.56 miles
Jada lives in Town B. She tells a friend from another state how far away she lives from Town A.	204 miles
A mapping company is making an accurate map of the area.	200 miles

Solution

A **road sign** wouldn't give distances to the nearest hundredth of a mile.
The road sign would give the distance as **204 miles**.
This is accurate enough, and can be read easily by a passing driver.

Jada's friend would only want a **rough idea** of how far away Town A is.
Jada could tell her friend that she lives **200 miles** from Town A.
The exact distance **doesn't matter** to someone who lives far away.

The **mapping company** needs to know **exact distances** to draw an **accurate map**. They would use the figure of **203.56 miles**.

Guided Practice

Exercises 1–4 give three figures: one exact and two rounded numbers. Choose which you think is most suitable, and explain your choice. (In each case, the first figure given is the exact answer.)

1. Ana Lucia writes in a history essay: Thomas Jefferson lived to the age of 83 / 80 / 100.

2. On a form, Gavin gives his height as 160.67 cm/161 cm/200 cm.

3. A school records how many students are in school each day. Today there are 3914/3900/4000 students attending.

4. Mr. Anderson returns from a vacation in England with £10 left over from his extra cash. £1 is worth $1.85965. He changes the money at the bank and receives $18.5965/$18.60/$19.

The Amount of Rounding Affects the Accuracy

If you use rounding to **estimate** a **sum**, be careful how much you round. Rounding to **higher place values** usually gives an estimate farther from the **actual answer** than rounding to **lower place values**.

Check it out:
Using rounded numbers to check your answer won't ever tell you that your answer is definitely right, only whether it is reasonable.
Your answer might be close to the real answer but could still be wrong.

Example 2

Lucas wants to add 3439 and 5482. He doesn't need an exact answer, so he decides to use rounding. Look at Lucas's work below.
How could he have found a more accurate answer?

$$\begin{array}{r} 3000 \\ +\ 5000 \\ \hline 8000 \end{array}$$

Solution
Lucas rounded to the nearest **thousand**, so he got an estimate of **8000**.
If he had rounded to the nearest **hundred**, he would have gotten 3400 + 5500 = **8900**, which is much closer to the actual answer of **8921**.

Guided Practice

5. Estimate 962 – 246 by rounding to the nearest hundred.

6. Estimate 962 – 246 by rounding to the nearest ten.

7. Calculate 962 – 246 exactly.
Which of your two estimates was closer to the actual result?

Rounded Numbers Can Be Used to Check Work

Many times you'll want to **check your work** without doing the calculation all over again. **Rounding** is a way to see if your answer is **reasonable**.

Example 3

Calculate 2343 + 5077.
Then check your work by rounding to the nearest hundred.

Solution

Actual sum:

$$\begin{array}{r} 2343 \\ +\ 5077 \\ \hline 7420 \end{array}$$

Rounded sum:

$$\begin{array}{r} 2300 \\ +\ 5100 \\ \hline 7400 \end{array}$$

← These are rounded to the nearest hundred

The answer to the rounded sum is **close** to the answer to the actual sum, so the answer to the actual sum is **reasonable**.

Example 4

Martin is trying to solve 29.6 × 9.8. He gets the answer 192.08. Check Martin's answer by rounding to the nearest ten.

Solution

Rounded sum: $30 \times 10 = \mathbf{300}$

Martin's answer is a **long way** from the rounded answer, so it looks like his answer of 192.08 might be wrong.

In fact $29.6 \times 9.8 = \mathbf{290.08}$

This is **much closer** to the rounded estimate.

Guided Practice

In Exercises 8–15, check the answers given by rounding to the place value shown in parentheses. Say whether the answers are reasonable.

8. 1818 + 700 = 3918 (hundred)
9. 22 × 79 = 738 (ten)
10. 490 + 770 = 1260 (hundred)
11. 642 – 369 = 273 (ten)
12. 2.85 × 52.1 = 96.385 (one)
13. 68 × 47 = 5032 (ten)
14. 32.815 + 84.565 = 117.38 (ten)
15. 10.48 × 67.02 = 902.3696 (one)

Independent Practice

In Exercises 1–6, use rounding to check the answers given, and say whether or not you think they are reasonable:

1. 6898 + 517 = 7415
2. 97 × 411 = 13,677
3. 547 × 695 = 527,855
4. 74,861 – 2940 = 65,621
5. 96 × 7973 = 526,218
6. 4362 – 1855 = 2507

7. Darnell is trying to work out the answer to 52 + 871. Darnell thinks the answer is 923. He asks Zoe and Enrique if his answer looks about right. Zoe thinks the answer should be near to 1000. Enrique thinks the answer should be about 920. Why might Zoe and Enrique have gotten different answers?

8. In the Olympic 100 meters final, the first three athletes finish the race in times of 9.89 seconds, 9.94 seconds, and 9.99 seconds, to the nearest hundredth of a second. Why would it not be a good idea to round these times any further?

9. Town C is putting up a new sign showing its population. The population is 45,691 people, but this figure is changing all the time so the town decides to use a rounded figure. What figure do you think they should use: 45,690, 45,700, 46,000, or 50,000? Explain your answer.

Now try these:
Lesson 1.4.2 additional questions — p430

Round Up

Remember that a rounded number is usually not the same as the real figure.
It only gives you a guide to how big the real number is.

Lesson 1.4.3

Estimation

Estimation means "making a good guess."
You can use it if you don't need to know an exact answer, or if a question has no exact right answer.

You Can Estimate When There's No Exact Answer

Sometimes in math there is **no exact right answer**.
You can use the information you do have to make an **estimate**.

Example 1

Carla has a tall bookshelf and a short bookshelf. When full, the tall bookshelf can hold about 60 books. Estimate from the picture how many books the small bookshelf will hold.



Solution

There is no exact number of books you can fit on a bookshelf, because not all books are the same size.

To **estimate** the answer, **compare** the bookshelves. The tall one has **3 shelves**, and the small one only **2**. All the shelves are the same size, so the small bookshelf will hold **around two-thirds** the number of books.

So you can estimate that the small bookshelf will hold **about 40 books**.

California Standard:

Mathematical Reasoning 2.3 **Estimate unknown quantities graphically and solve for them by using logical reasoning and arithmetic and algebraic techniques.**

What it means for you:

You'll practice working out an approximate answer when you can't find, or don't need, an exact one.

Key words:

- estimate
- compare
- exact answer

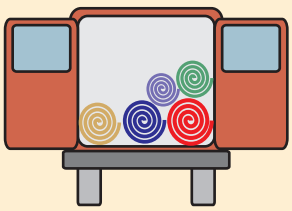
Check it out:

Use rounding to work out rough amounts — see Lessons 1.4.1 and 1.4.2 for more on rounding.

Guided Practice

1. This crate is partially filled with 18 apples. Estimate the total number of apples the crate can hold.





2. Mr. Lawrence is loading rolls of carpet of varying thickness into his van.
He has put 5 in so far. Estimate how many rolls he will get in the van in total.

3. Jermaine collects seashells, and gets two new cases to display some of them. He has filled the smaller one with 19 shells. Estimate how many will fit in the larger case.

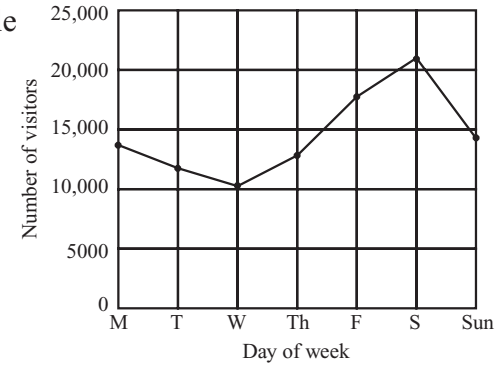


You Can Estimate If You Don't Need an Exact Answer

You don't always **need** to use an **exact figure**.
Sometimes an **estimate** is enough.

Example 2

This graph shows how many people visited a theme park last week. The manager only wants a rough idea of how many visitors there were each day.



Estimate the total number of people who visited the park over Friday and Saturday.

Solution

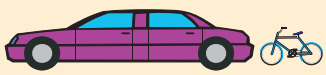
The value for Friday is **about halfway** between **15,000** and **20,000** on the vertical axis, so a **good estimate** for Friday would be about **17,500 people**. A good estimate for Saturday might be **21,000**, because the value for Saturday is **just above 20,000**.

This gives an estimate of 21,000 + 17,500 = **38,500** for the total number of people visiting the park over Friday and Saturday.

Guided Practice

4. Use the graph from Example 2 to estimate the number of theme park visitors on Monday, Tuesday, Wednesday, Thursday, and Sunday.

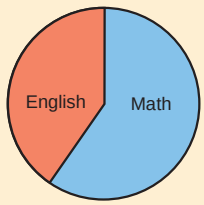
5. This car is 20 feet long. Estimate the length of the bicycle.

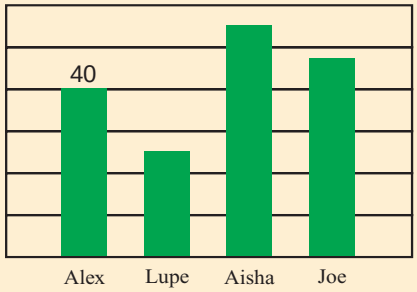


6. The house in this picture is 30 feet tall. Estimate the height of the tree.



7. Carlos spent a total of 50 minutes doing his English and math homework. The circle graph shows how much of the time he spent on math and how much on English. Estimate how many minutes Carlos spent doing his math homework.





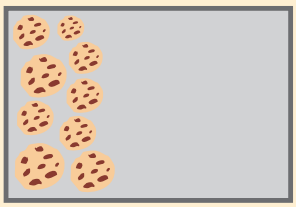
8. This bar graph shows the number of points scored by four students in a math test. Alex scored 40 points. Estimate the number of points scored by Lupe, Aisha, and Joe.

Independent Practice

1. George has two sunflowers in his garden, shown in the picture.

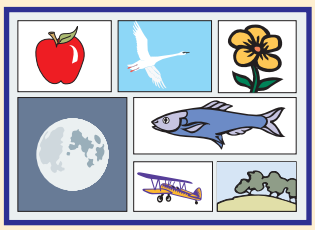
The taller flower is 62 inches tall.
Estimate the height of the shorter sunflower.





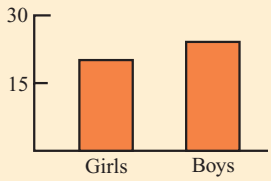
2. Casey is baking cookies.
She has put 9 cookies on the cookie sheet as shown. Estimate the total number of cookies Casey can fit on this tray.

3. Ms. Marquez is putting up pictures the students in her art class have painted. Her display boards are shown below. She has put up 7 pictures on the smaller board. Estimate how many pictures she can fit on the larger board.





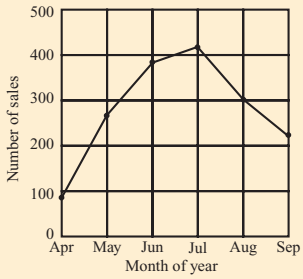
The bar graph shows the number of 6th grade boys and girls who went on a trip to the museum.
4. Estimate how many girls went to the museum.
5. Estimate how many boys went to the museum.





6. The circle graph shows how many soccer balls, footballs, and basketballs a sports store has in stock. The total number of balls in stock is 60.
Estimate how many of each type of ball are in stock.

The graph below shows the sales of sunglasses from one store for six months of the year.
Use it to answer Exercises 7–9.
7. Estimate the number of pairs of sunglasses sold in April.
8. Estimate the total number of pairs of sunglasses sold in June, July, and August.
9. Estimate the average monthly sales for the period.



Now try these:
Lesson 1.4.3 additional questions — p431

Round Up

Some estimates are better than others, but remember there's often no exact answer when you're giving an estimate. Make sure you always check your answer to make sure the estimate is reasonable.

Lesson 1.4.4

Using Estimation

Estimation is really useful in a lot of real-life situations, where you might not be able, or don't need, to do an exact calculation. There are other times when it's better to figure out the exact answer.

Estimates Aren't Always a Good Idea

There are some situations where you definitely **shouldn't** use an estimate.

Example 1

Dr. Brown is giving medication to a patient. He needs to give 1.5 milligrams of the medicine for every pound that the patient weighs. Is it appropriate for Dr. Brown to estimate the amount of medicine for a particular patient?

Solution

Dr. Brown needs to make **precise** measurements of the amount of medicine — it's very important that he gives the correct dose.

Guided Practice

In Exercises 1–7, say whether each situation needs a precise figure, or if an estimate would be more suitable, and give a reason for your answer:

1. Mr. Bishop is deciding how much gas to put in his car at the start of a long journey.

2. Mrs. Suarez is figuring out how much wallpaper she needs for her bedroom.

3. Sasha's pumpkin is weighed for the annual "heaviest pumpkin" competition.

4. Professor Elliott is finding the heights of a class of children for a scientific study.

5. Peter is calculating how many wins and losses his baseball team had this season.

6. Susie is deciding how many sandwiches to make for a party.

7. Ms. Ryan is figuring out the grade point averages of the students in her class.

California Standards:

Mathematical Reasoning 2.1 Use estimation to verify the reasonableness of calculated results.

Mathematical Reasoning 2.6 Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.

What it means for you:

You'll learn when it's a good idea to use estimation in calculations, and when it's not. You'll also learn about another type of estimation, called front-end estimation.

Key words:

- estimate
- precise
- front-end estimation

Front-End Estimation Is Another Kind of Rounding

Another type of estimate is **front-end estimation**. To do this, you add the **first digit** in each number, and then make a rough **estimate** of the value of the remaining digits.

Check it out:
Front-end estimation is most often used to figure out prices. It's a type of estimation that people often do in their heads in stores and restaurants.

Example 2

Manuel is standing in line in the cafeteria. He wants to buy a sandwich, some soup, and a salad, but he only has a $10 bill.

Use front-end estimation to estimate how much Manuel's lunch will cost.

MENU

(All prices include tax.)

Sandwich	$4.49
Salad	$1.79
Soup	$3.69
French fries	$1.10
Soda	$1.29
Water	$1.15

Solution

Manuel wants to estimate $4.49 + $3.69 + $1.79.

Add the **front-end** digits:

$$\begin{array}{r} \mathbf{4}.49 \\ \mathbf{3}.69 \\ +\ \mathbf{1}.79 \\ \hline \mathbf{8} \end{array}$$

Then estimate **the rest**:

$$\begin{array}{r} 4.\mathbf{49} \\ 3.\mathbf{69} \\ +\ 1.\mathbf{79} \\ \hline \text{about } \mathbf{2.00} \end{array}$$

And **add** the two parts:

$$\begin{array}{r} \mathbf{8.00} \\ +\ \mathbf{2.00} \\ \hline \mathbf{10.00} \end{array}$$

Front-end estimation gives an estimated cost of **$10** for Manuel's meal.

He should calculate the exact total to see if he can buy the 3 items. $4.49 + $3.69 + $1.79 = $9.97, so he **can** buy all 3 items — but the estimate was too close to be sure.

Guided Practice

In Exercises 8–12, use the menu shown in Example 2. Estimate each total using front-end estimation.

8. A sandwich, a bowl of soup, and a water.

9. A salad and two bags of French fries.

10. A bag of French fries, a soda, and two salads.

11. Two bowls of soup, two bags of French fries, and two sodas.

12. A salad, a bowl of soup, two sandwiches, three bags of French fries, and a water.

13. Which of the sums from Exercises 8–12 would you calculate exactly to see if you could afford them if you only had $10?

Independent Practice

In Exercises 1–6, say whether each situation needs a precise figure, or if an estimate would be more suitable, and give a reason for your answer:

1. Dr. Burke is taking the temperature of a patient with a fever.

2. Ms. Bennett is figuring out how long it will take her to grade the homework from her math class.

3. Clifton is weighing his suitcase before he goes on vacation to make sure he doesn't have to pay for excess baggage.

4. The police tell news reporters the height of a suspect based on descriptions from witnesses.

5. A biologist is figuring out the size of the population of wild rabbits in California.

6. A butcher is figuring out how much to charge for 4 pounds of steak.

7. Max wants to buy 3 books, priced $7.95, $6.45, and $8.88. Estimate the total cost of the books using front-end estimation.

8. Julieta wants to buy a scarf for $9.29, a pair of gloves for $8.59, and 2 hats for $5.85 each. Use front-end estimation to estimate how much money she will need.

9. Mr. Benitez owns a pet store. He aims to make $1000 every 3 days. The receipts for the last 3 days came to $370, $365, and $397. Estimate the total receipts for these 3 days by front-end estimation.

10. Shantice is eating in a restaurant. Her appetizer costs $6.95, the main course is $15.20, and the dessert costs $9.65. Use front-end estimation to find the approximate cost of Shantice's meal.

To the right is a price list of items from a sporting goods store. Use front-end estimation to estimate the cost of the following combinations of items:

Baseball bat	$46.50
Baseball	$11.49
Basketball	$32.69
Football	$52.99
Football helmet	$89.99
Hockey stick	$63.49

11. A football and a football helmet.
12. A hockey stick and a baseball bat.
13. A baseball, a basketball, and a football.
14. 2 baseball bats and 2 baseballs.
15. One of everything on the list.

Now try these:
Lesson 1.4.4 additional questions — p431

Round Up

Estimating can be a really useful method, but you shouldn't use it all the time. You always need to decide whether estimating will give you an answer that's useful for solving the problem — and sometimes you need to figure out the precise solution even after you've tried doing an estimation.

Chapter 1 Investigation
Populations

You've been asked to write a report about the population of nine different counties in California. The table below contains some data you will find useful — it shows the population of the nine counties on a particular day in two different years.

Part 1:
You have been asked to include some specific information in your report:

- Lists of the counties **in order of population**: (i) in **2004** (ii) in **2005**.
- A list showing the counties' **population change** between 2004 and 2005.
 Include comments on which counties' populations **increased** and **decreased** most.
- A **bar graph** showing the populations in each year. Before plotting the bar graph, each figure must be **rounded** to the **nearest hundred**. The vertical axis of your graph should be labeled from 0 to 40, and must show the populations in "**numbers of thousands**."

County	Population (2004)	Population (2005)	Number of births
Alpine	1,274	1,242	11
Amador	37,552	38,221	271
Lassen	35,325	35,696	312
Modoc	9,828	9,813	92
Colusa	20,663	21,315	359
Plumas	21,378	21,557	181
Mono	13,529	13,512	179
Sierra	3,537	3,514	19
Trinity	13,826	14,025	117

Part 2:
If the **number of births** in each county remained the **same** for the next 10 years, and nothing else happened to cause a change in the population, what would the population of each county be in: (i) **2008** (ii) **2015**?

Do you think these predictions are **realistic**? Explain your answer.

Extension

- Order all nine counties in order of their **numbers of births**, from least to greatest. How does this list compare to the lists that you obtained in Part 1? ← Describe some similarities and differences.
- A county's **birthrate** is the number of births per thousand people.
 To find it, divide the number of births by the population (in 2005), and then multiply the result by 1000. How does the list of birthrates compare to the list of numbers of births?

Open-ended Extension

Write the report on the populations in the nine counties.
You may need to do some further research.

- You should include all the information you've found out so far.
- Include any other information or graphs you think are useful.
- You may also extend the report to include information about other counties.

Round Up

The number skills you used in this Investigation are ones that you may use every day for the rest of your life. For example, when you add up prices in a store, these are exactly the skills you use.

Chapter 2

Expressions and Equations

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Section 2.1 introduction — an exploration into:

Algebra Tile Expressions

You can represent values using blue and red tiles — blue tiles represent positive values, while red tiles represent negative values. But it's not just numbers like 3, 4, or –5 that you can show. Tiles can also be used to show more complicated expressions involving variables — such as $x + 5$.

In this Exploration, **blue square** tiles have a value of **+1**, and **red square** tiles have a value of **–1**. You can use these tiles to represent **numbers**, like 3, 4, or –5.



Variables are things whose **value** you **don't know**. They are usually given names like ***x*** or ***y***. These funny-shaped tiles show the **variables *x*** and ***–x***.

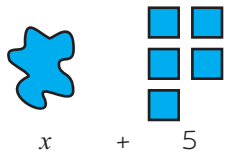


Example

Show the expression $x + 5$ with algebra tiles.

Solution

The **blue shape** represents the **variable *x***, and the **5 blue tiles** represent **+5**. Together the group of tiles shows the expression ***x* + 5**.

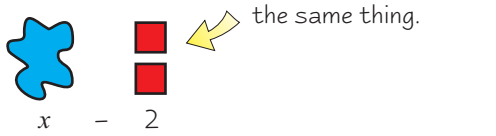


Example

Show the expression $x - 2$ with algebra tiles.

Solution

The **blue shape** represents the **variable *x***, and the **2 red tiles** represent **–2**. Together the group of tiles shows the expression ***x* – 2**.

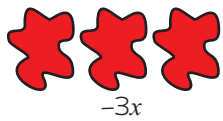


Example

Show the expression $-3x$ with algebra tiles.

Solution

The expression $-3x$ means the same as "3 groups of $-x$," or $(-x) + (-x) + (-x)$. So the tiles on the right show the expression ***–3x***.

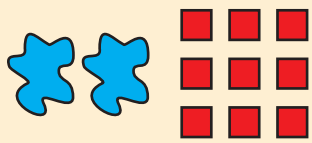


Exercises

1. Use algebra tiles to model each of these expressions:

a. $x + 7$ **b.** $x - 4$ **c.** $-x + 2$
d. $4x$ **e.** $-2x$ **f.** $2x - 1$

2. Write the expression represented by the diagram.



Round Up

In math, expressions with variables are really useful because you can use them to represent just about anything. The next few Lessons show you how to write down many different amounts as mathematical expressions.

Lesson 2.1.1

Section 2.1 Variables

One difficulty with math problems is that you can't get started if you're not given all the numbers for a problem. For instance, you can't work out the area of a rectangle unless you're given the width and height. That's where variables come in.

California Standard:

Algebra and Functions 1.2

Write and evaluate **an algebraic expression for a given situation, using up to three variables.**

What it means for you:

You'll learn a way of using letters or symbols in place of numbers that you don't yet know.

Key words:

- variable
- unknown

A Variable Is an Unknown Amount

Sometimes you need to write out a math problem but you don't know all the numbers. All you need to do is write a **letter** or **symbol** to represent each **unknown number**. The letters or symbols are called **variables**.

You can use **any** letter or symbol as a variable — but the most common variable is ***x***.

Example 1

You are trying to write out a math problem that involves the length of a line. However, you don't know the length of the line. What can you do?

Solution

You can replace the number you don't know with a **variable**.

You can use whatever letter or symbol you like to represent the unknown length. For example, you might choose ***x***.

You can write the problem out with an ***x*** where the length of the line should be.

Example 2

Mr. Harris has $7.50 more than Mr. Wilson. What extra information do you need to find out how much money Mr. Harris has?
How could you represent this unknown variable?

Solution

You need to know **how much money Mr. Wilson has**. In the problem, you're not told how much Mr. Wilson has, so it's an **unknown amount**.

That means you need a **variable** to represent "the amount of money Mr. Wilson has." Pick a variable that will be easy to remember, and then write a sentence like:

Let *w* represent the amount of money Mr. Wilson has.

Check it out:

Try to choose sensible letters for variables — letters that will make it easy to remember what they represent.
For example, w is a good letter to use for the amount of money owned by Mr. Wilson. But r would be better if the person was Mrs. Richards. This is especially important if you need to use more than one variable — see Lesson 2.1.3.

Guided Practice

Identify an unknown in Exercises 1–4 that could then be used to find the required quantity. Choose a letter to represent this quantity.

1. Emilio and James started doing their homework at the same time. Emilio finished 10 minutes after James. You want to know how long Emilio spent on his homework.

2. Julianne finished a test at 2:45 p.m.
You want to know the time when she started the test.

3. Mr. Mason's truck weighs 4 times as much as his car. You want to know the weight of Mr. Mason's truck.

4. Adam finished the race 2 seconds before Juan. You want to know how long it took Adam to finish the race.

Variables Work Just Like Normal Numbers

You can **add** and **subtract** with variables just like you would with normal numbers.

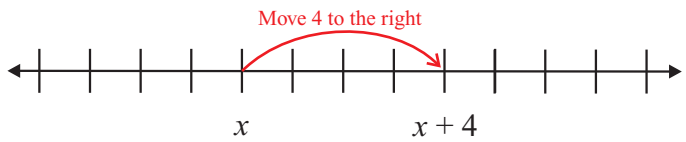
Check it out:
This is exactly the same kind of picture that you saw in Section 1.1 — only with variables instead of numbers.

Example 3

4 is added to the variable, x. Show this on a number line.

Solution

To add 4 to x, start at x and then move **4 to the right**. It doesn't matter that you don't know exactly where x is on the number line. You just need to show what you would do if someone told you what x represents.

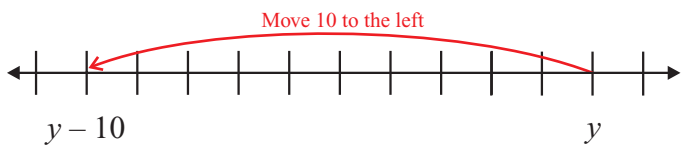


Example 4

10 is subtracted from the variable, y. Show this on a number line.

Solution

To subtract 10 from y, start at y and then move **10 to the left**.



Don't forget:
Look at the diagrams in Section 1.1 if you're not sure what these should look like.

Guided Practice

Show the sums in Exercises 5–16 on number lines.

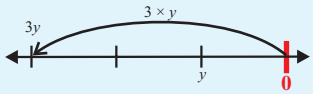
5. $x + 2$	**6.** $x - 3$	**7.** $x - (-3)$
8. $y + (-5)$	**9.** $y - 9$	**10.** $y + 14$
11. $z + (-15)$	**12.** $z + 23$	**13.** $z - (-2)$
14. $a + 3 - 1$	**15.** $a + 6 + 5$	**16.** $a - 9 - (-7)$

You Can Multiply and Divide with Variables Too

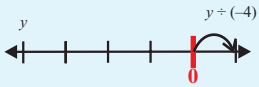
When you **multiply** or **divide** with variables, you could be multiplying or dividing by either **positive** or **negative** numbers.

This might seem to make things complicated, but you just have to remember that the variable will still behave like **any other number**.

Check it out:
In this example, the variable y is a positive number. You need to remember that if the variable is negative, the arrows on the number lines would all point in the opposite direction.
For example, if y is negative, $3 \times y$ looks like this:



$y \div (-4)$ looks like this:

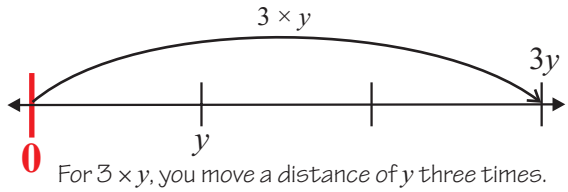


Example 5

Use a number line to show the following operations with the positive variable y:
y multiplied by 3
y multiplied by -3
y divided by 4
y divided by -4

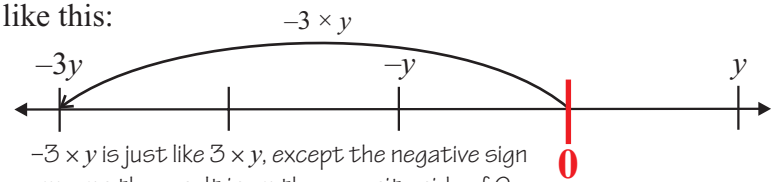
Solution

$3 \times y$ looks like this:



For $3 \times y$, you move a distance of y three times.

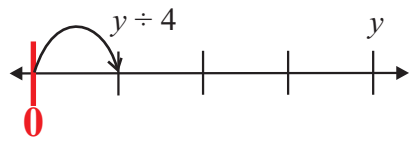
$-3 \times y$ looks like this:



$-3 \times y$ is just like $3 \times y$, except the negative sign means the result is on the opposite side of 0.

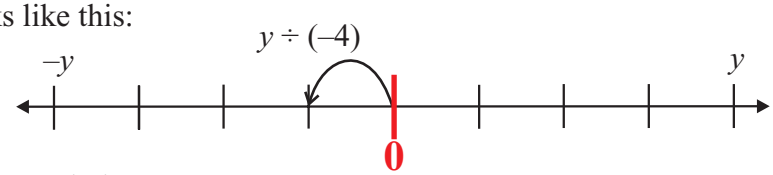
Check it out:
Notice how $-3 \times y$ is the same as $3 \times (-y)$.
And $y \div (-4)$ is the same as $-y \div 4$.

$y \div 4$ looks like this:



For $y \div 4$, you move a fourth of the distance to y.

$y \div (-4)$ looks like this:



$y \div (-4)$ is just like $y \div 4$, except the negative sign means the result is on the opposite side of 0.

Guided Practice

Don't forget:
Look at the diagrams in Example 5 if you're not sure what these should look like.

Represent these calculations on number lines. Do each problem twice — once for a positive variable, and once for a negative.

17. $q \times 6$ **18.** $q \times 5$ **19.** $q \times (-2)$
20. $r \div 3$ **21.** $r \div (-3)$ **22.** $r \div 5$
23. $s \times 2$ **24.** $s \times (-5)$ **25.** $s \div 1$

Independent Practice

Now try these:
Lesson 2.1.1 additional questions — p431

Identify an unknown in Exercises 1–9 that could then be used to find the required quantity. Choose a letter to represent this unknown.

1. Karen eats 5 more cookies than Lupe.
You want to know how many cookies Karen has eaten.

2. Laura and Kenny set up lemonade stands.
On the first day, Laura sells five more cups of lemonade than Kenny.
You want to know how many cups of lemonade Laura sold.

3. Deandre gets 36% more than Ana on a history test.
You want to know Deandre's score.

4. Madison measures her arm with a piece of string.
It is exactly 5 times as long as the string.
You want to know the length of Madison's arm.

5. Gregory has 25 cakes. He shares them equally between his friends.
You want to know how many cakes each friend gets.

6. Carolyn takes one-fourth as long as Nina to do her homework.
You want to know how long Nina takes.

7. Tanya is selling some old toys that she no longer plays with and Imelda buys half of them. You want to know how many toys Imelda bought.

8. A rectangle has one side of length 5 inches.
You want to know the area of the rectangle.

9. Tim has one-fourth of the amount of money that he had last week.
You want to know how much money Tim has.

Show the sums in Exercises 10–15 on a number line.

10. $h + 7$ **11.** $h - 1$ **12.** $h - 2$
13. $m + (-5)$ **14.** $m - (-3)$ **15.** $m + 3$

Draw the calculations in Exercises 16–21 on the number line. Show the cases both where the variable is positive, and where it is negative.

16. $v \times 5$ **17.** $v \times 2$ **18.** $v \times (-3)$
19. $w \div 9$ **20.** $w \div 3$ **21.** $w \div (-4)$

Round Up

At first it can look kind of confusing if you see a sum with letters in it — but just remember that the letters are just standing in for unknown numbers.

Lesson 2.1.2

Expressions

California Standard:

Algebra and Functions 1.2 **Write and evaluate an algebraic expression for a given situation**, using up to three variables.

What it means for you:

You'll practice how to make your own expressions, and how to work out the value of an expression once you're told the values of the variables.

Key words:

- variable
- expression
- evaluate

Check it out:

Mathematical expressions can be numerical or algebraic. Numerical expressions only contain numbers — for example, $2 + 4$, $3 \times 4 - 7$, and so on.
Algebraic expressions contain at least one variable — for example, $x + 4$, $3x - 7$.

Variables are all well and good, but they're only useful when you use them to solve math problems. You can use variables and numbers to describe a problem in math terms — it's called an expression.

An Expression Is a Problem Written in Math Terms

Expressions are used to write word problems in math terms. Expressions are like **instructions** that tell you what you have to do to a number or variable.

In Words	Expression
A number, x, is increased by 7	$x + 7$
A number, y, is decreased by 7	$y - 7$
A number, a, is multiplied by 7	$a \times 7$
A number, k, is divided by 7	$k \div 7$

Sometimes you might have to describe a **real-life situation** using a mathematical expression.

You need to imagine what would happen to a quantity, and write that down using variables, and $+$, $-$, $\times$, and $\div$.

Example 1

Peter has \$165 in his savings account. Lottie has x more dollars than Peter. Write an expression for the amount of money that Lottie has.

Solution

Lottie has more money than Peter. To find how much she has, you need to **add \$$x$** to the amount Peter has. So the expression for the number of dollars Lottie has is **$165 + x$**.

Guided Practice

In Exercises 1–6, write an expression for each word problem.

1. A number, x, is decreased by 10.
2. A number, q, is increased by 27.
3. The variable w is multiplied by 20.

Don't forget:
You have to calculate things in parentheses before any other parts of the expression.

4. A computer repairman charges $25 per hour to fix computers. How many dollars should he charge for h hours of work?
5. Aisha's jump rope is 2 feet shorter than Lisa's. If y represents the length of Lisa's rope in feet, then write an expression to represent the length of Aisha's rope (also in feet).
6. Describe the expression $100 - (4 \times z)$ in your own words.

You Can Evaluate if You Know the Value of the Variable

If you're given a value to use for a variable, you can work out the value of the expression. It's called **evaluating the expression**.

Example 2

Evaluate the expression $7 \times h$, if the value of h is 6.

Solution

Start with $7 \times h$.

But $h = 6$, so replace h with the number 6: **7×6**

This means evaluating the expression gives **$7 \times 6 = 42$**

Example 3

The expression $120 - s$ is used to find out how many seats are available in a movie theater after s people are already seated. How many seats are available if 73 people are already seated?

Solution

Start with $120 - s$.
You're told that $s = 73$, so replace s with the number 73

This means that **$120 - 73 = 47$ seats** are available.

Guided Practice

Evaluate the expressions given in Exercises 7–12.

7. $t + 28$, given that $t = 19$
8. $y - 7$, given that $y = 5$
9. $d \times 9$, given that $d = 3$
10. $g \div 5$, given that $g = 25$
11. $x + (-15)$, given that $x = 13$
12. $c - (-3)$, given that $c = 10$

13. The expression $15 + t$ gives Earl's height in centimeters, where t is his friend Tony's height in cm. If Tony is 135 cm tall, how tall is Earl?

14. The amount of money made by a hot dog stand is $\$2.5h$, where h is the number of hot dogs sold. How much money does the stand make if it sells 271 hot dogs in one day?

There Are Three ways of Writing Multiplication

In an expression, there are three ways of writing a multiplication. The first is to use the **× sign**.

Sometimes a **raised period, •**, is used instead of ×.

If you're just multiplying a **variable** by a number, then you can **leave out the symbol** completely.

$4 \times p$

$4 \bullet p$

$4p$

They all mean the same thing

Check it out:

Watch out — the • can be a bit confusing because it looks like a decimal point. The • sign just means "multiplied by."

Guided Practice

Rewrite Exercises 15–18 using • as a multiplication symbol.

15. $5 \times i$ **16.** $12 \times t$ **17.** $8r$ **18.** $-9k$

Rewrite Exercises 19–22 using × as a multiplication symbol.

19. $11 \bullet d$ **20.** $4 \bullet q$ **21.** $4q$ **22.** $53z$

Rewrite Exercises 23–26 without a multiplication symbol.

23. $7 \bullet w$ **24.** $13 \times t$ **25.** $-3 \times y$ **26.** $4 \bullet c$

Independent Practice

In Exercises 1–6, evaluate the expression $19 - y$ for each value of y.

1. $y = 11$ **2.** $y = -11$ **3.** $y = 26$

4. $y = 19$ **5.** $y = 0$ **6.** $y = -8$

7. A variable, d, is multiplied by 15. Write this as an expression. Evaluate the expression for $d = 4$.

8. One palm tree is sixteen feet taller than another. Write an expression for the height of the shorter palm tree.

9. Jessica shares q potato chips between 12 friends. Write an expression for the number of potato chips received by each friend. Evaluate the expression if $q = 96$.

Exercises 10 and 11 are about Manuel, the manager at a restaurant.

10. Manuel collected t dollars in tips and wants to share it equally among his n employees. Write an expression that can be used to calculate how much each employee will get in tips. You can assume that Manuel does not get any tips.

11. If there are 5 employees and Manuel collected $90 in tips, use your expression from Exercise 10 to calculate how much each employee will get.

Rewrite Exercises 12–14 to show all three ways of writing multiplication.

12. $3 \times w$ **13.** $7 \bullet t$ **14.** $-9r$

Now try these:

Lesson 2.1.2 additional questions — p432

Round Up

To evaluate an expression, you just need to substitute numbers in place of variables. They take some getting used to, but expressions are a really good way to write down and then solve math problems.

Lesson 2.1.3

Multi-Variable Expressions

California Standard:
Algebra and Functions 1.2
Write and evaluate an algebraic expression for a given situation, using up to three variables.

What it means for you:
You'll see expressions containing two or three variables.

Key words:
- variable
- expression
- evaluate
- term

Check it out:
Remember these key words:
sum — this means you have to add two or more numbers
difference — this means you have to subtract one number from another
product — this means you have to multiply two or more numbers
quotient / **ratio** — both these words mean you have to divide one number by another

Don't forget:
If part of an expression is written in parentheses, you should always work out that part first.

Some expressions contain more than one variable — you've met some of those before but there are plenty more in this Lesson. You'll also get some more practice substituting values for variables.

Some Expressions Use Two Variables

If there are **two numbers** you don't know, that's not a problem. You can use **two** different variables, one for each unknown number.

In Words	Expression
The sum of a and b	$a + b$
The product of v and w	$v \times w$, $v \cdot w$, or vw
p is subtracted from $9q$	$9q - p$

You can use expressions with two (or more) variables to represent situations with more than one unknown quantity.

Example 1

DeAndre, Kia, and Anthony each thought of a number. DeAndre's number was twice as big as the sum of Kia and Anthony's numbers. Write an expression for DeAndre's number.

Solution

Represent Kia's number by k, and Anthony's number by a.

Then DeAndre's number is $\mathbf{2 \times (k + a)}$.

Evaluating expressions with two or more variables is the same as for one variable. Just **substitute** the numbers you're given and **find** the result.

Example 2

Evaluate the expression $x - y$ for $x = 20$ and $y = 8$.

Solution

Replace x with 20 and y with 8.
That means that $x - y$ becomes $\mathbf{20 - 8}$, so the answer is $\mathbf{12}$.

Example 3

Evaluate the expression $(15 - f) \times (7 + z)$ for $f = 2$ and $z = 3$.

Solution

Replace f with 2 and z with 3.

The expression becomes $\mathbf{(15 - 2) \times (7 + 3) = 13 \times 10 = 130}$.

Guided Practice

In Exercises 1–4, evaluate the expressions for the values given.

1. $e + f$ when $e = 5$ and $f = 7$

2. $q \div s$ when $q = 18$ and $s = 3$

3. $y - (5t)$ when $y = 10$ and $t = 1$

4. $(k \times 9) + z$ when $k = 11$ and $z = 6$

Write an expression for each of Exercises 5–7.

5. w is divided by v.

6. r is multiplied by t.

7. k is multiplied by 3, t is divided by 4, then the results are added together.

Some Expressions Use Three Variables

If a calculation contains **three unknowns** then you'll need to include **three variables** in your expression.

Example 4

Wesley has w cookies and Maribel has m cookies.
They decide to eat c cookies between them.
Write an expression for the total number of cookies left.

Solution

The number of cookies Wesley and Maribel have before they eat any is

$$w + m$$

So once they have taken away c by eating them, there will be

$$w + m - c$$

Guided Practice

In Exercises 8–12, evaluate the expression $z + y - x$ for the values given.

8. $x = 5, y = 5, z = 2$

9. $x = 1, y = 1, z = 1$

10. $x = 2, y = 3, z = -2$

11. $x = 9, y = -3, z = 1$

12. $x = 2, y = 0.5, z = 3$

13. An elevator leaves the first floor and goes up x floors.
The elevator then goes down y floors, then goes up another z floors.
Write an expression for the floor that the elevator is on now.

14. The amount that Mr. Jackson is paid each week is calculated by first multiplying his hourly wage, a, by the number of hours he worked that week, b. Then the amount, c, that he owes in taxes is subtracted from this amount. This is represented by the expression $ab - c$. Use the chart on the right to calculate the amount of Mr. Jackson's paycheck this week.

hours worked:	24
hourly wage:	\$9.10
taxes:	\$39.80

Check it out:

In an expression without parentheses such as $ab - c$, always carry out the multiplication before the addition.
So here, multiply a by b, then subtract c from the result.

Expressions Are Made Up of Terms

Check it out:
In an expression, any quantity that isn't a variable is called a constant.

Parts of an expression that are separated only by + or – are called **terms**. Each "+" or "–" belongs to the term that follows.

Check it out:
"+ n" is the same as "n."

Example 5

What are the terms in the expression $3 \times l - 2 \times m + n - 7$?

Solution

Terms are separated by + or – signs, so the terms are:

(i) $3 \times l$, (ii) $-2 \times m$, (iii) n, and (iv) -7.

When you evaluate **multi-variable expressions** with more than one term, it makes sense to evaluate **each term** individually first.

Check it out:
Evaluating terms individually (and writing down the results) also makes it easier to find any mistakes you make along the way.

Example 6

Evaluate $5a + 3b + 2c$, when $a = 2$, $b = 3$, $c = 5$

Solution

First, you need to put the values of a, b, and c into the expression:

$$(5 \times 2) + (3 \times 3) + (2 \times 5)$$

You can now evaluate each term:

$$10 + 9 + 10$$

Finally, you can evaluate the simplified sum:

$$10 + 9 + 10 = 29$$

Guided Practice

Identify the terms in Exercises 15–20.

15. $v - 16$ **16.** $d + 22$ **17.** $f + (-g)$

18. $x - (-2) + y$ **19.** $m - 14 + 2n$ **20.** $13b + 5c - d$

In Exercises 21–26, evaluate the given expressions by first evaluating the terms.

21. $r + 4s + 3t$, when $r = 1$, $s = -4$, $t = 7$

22. $3r - 3s + 5t$, when $r = 5$, $s = 8$, $t = 2$

23. $5r + 10s - 2t$, when $r = 6$, $s = -3$, $t = 15$

24. $r + s + st$, when $r = 2$, $s = 9$, $t = 5$

25. $6 + rs + 7t$, when $r = 10$, $s = 11$, $t = 0$

26. $200 - 4rs - 3st - 2rt - rst$, when $r = 5$, $s = 1$, $t = 7$

Independent Practice

Evaluate the expression xy using the values of x and y given in Exercises 1–6.

1. $x = 0.5$ and $y = 20$
2. $x = -6$ and $y = -3$
3. $x = 0$ and $y = 5$
4. $x = -1$ and $y = 1$
5. $x = -10$ and $y = 10$
6. $x = 0.75$ and $y = 4$

Now try these:
Lesson 2.1.3 additional questions — p432

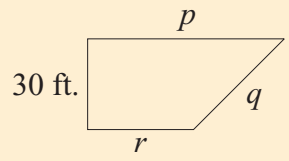
Exercises 7 and 8 are both about the same Math Club.
7. The Math Club had \$160 in its treasury. The club raised another j dollars from selling hot dogs after school and k dollars from selling snacks during school. Write an expression that represents how much money the Math Club has now.
8. Use your expression from Exercise 7 to work out how much money the Math Club have if they raised \$25 selling hot dogs and \$45 selling snacks.

Exercises 9 and 10 are both about Mrs. Roca.
9. Mrs. Roca took \$50 with her to the movies. She spent x dollars on tickets and y dollars on food. Write an expression that represents how much money Mrs. Roca had after the movie.
10. She spent \$18 on tickets and \$10 on food. Use your expression from Exercise 9 to calculate how much money she had left over after the movie.

Write an expression for each of Exercises 11–15.
11. d is divided by g.
12. The sum of r, s, and t.
13. k is subtracted from the sum of w and p.
14. The product of x, y, and z.
15. $2a$ is added to the product of b and c.

Don't forget:
The perimeter is the distance all the way around a shape.

Exercises 16–17 use the diagram to the right, showing the field that will be used to keep a horse.
16. Write an expression for the perimeter of the field.
17. Use your expression to calculate the perimeter of the field when $p = 42$ feet, $q = 38$ feet, and $r = 32$ feet.



18. Jeff is moving his desk, TV, and couch to his new apartment. His desk weighs x pounds, his TV weighs y pounds, and his couch weighs z pounds. He loads all three items into the back of his truck. His truck can carry up to 1000 pounds. Write an expression that represents how much more weight the truck can carry after the three items are loaded in the truck.

Round Up

No need to panic — when you're dealing with expressions that contain more than one variable, the rules are just the same as for single-variable expressions.

Lesson 2.1.4

Order of Operations

California Standards:

Algebra and Functions 1.3

Apply algebraic order of operations and the commutative, associative, and distributive properties **to evaluate expressions; and justify each step in the process.**

Algebra and Functions 1.4

Solve problems manually by using the correct order of operations or by using a scientific calculator.

What it means for you:

You'll see that it matters in which order you evaluate an expression, and you'll learn the correct order.

Key words:

- parentheses
- exponent
- multiply and divide
- add and subtract
- expression
- evaluate

Sometimes you'll meet expressions containing different combinations of operations (such as ×, ÷, +, and –). When you do, you need to be sure to do all of them in the right order.

The Order In Which You Evaluate Makes a Difference

The expression $2 + 3 \times 7$ looks like it could give two different answers, depending on how you work it out.

For example, working out "2 + 3" first gives: $2 + 3 \times 7 = 5 \times 7 = 35$, while working out "3 × 7" first gives: $2 + 3 \times 7 = 2 + 21 = 23$.

To avoid this situation, the following rule is used for all math expressions: **Always do multiplication before addition**, unless **parentheses** tell you to do otherwise.

Example 1

Evaluate $2 + 3 \times 7$, and $(2 + 3) \times 7$.

Solution

$2 + 3 \times 7 = 2 + 21 = 23$ — Work out the multiplication first

$(2 + 3) \times 7 = 5 \times 7 = 35$ — The parentheses tell you to do the addition first

In fact, you do all **multiplications and divisions before additions and subtractions** (unless **parentheses** tell you otherwise).

Example 2

Evaluate: (i) $32 - 3 \times 8$, (ii) $56 - 8 \div 2$.

Solution

(i) $32 - 3 \times 8 = 32 - 24 = 8$ — Work out the multiplication first

(ii) $56 - 8 \div 2 = 56 - 4 = 52$ — Work out the division first

You do multiplications and divisions working **from left to right**.
You also do additions and subtractions from left to right — after you've **finished** all the multiplications and divisions.

Example 3

Evaluate: $45 + 34 \div 17 - 2 \times 3$.

Solution

$45 + 34 \div 17 - 2 \times 3 = 45 + 2 - 2 \times 3$ — × and ÷ first, from left to right

$= 45 + 2 - 6$

$= 47 - 6 = 41$ — Then + and –, from left to right

Guided Practice

Evaluate the expressions shown in Exercises 1–8.

1. $4 + 2 \times 5$
2. $3 + 7 \times 4$
3. $12 \div 2 + 4$
4. $8 \times 7 + 1$
5. $11 - 1 \times 4$
6. $88 + 20 \div 2$
7. $(88 - 3) \div 5$
8. $8 + 9 \div 3 \times 5 - 3$

Don't forget:

Here, exponents means expressions like 4^5, y^2, x^3, and so on — they show repeated multiplication. The exponent is actually the little number written above and to the right of a larger number (or variable) called the base. The exponent shows the number of bases you need to multiply together. So $y^3 = y \times y \times y$.

Check it out:

An easier way to remember the order of operations is to remember the word **PEMDAS**, which stands for:
1. **P**arentheses
2. **E**xponents
3. **M**ultiplication / **D**ivision
4. **A**ddition / **S**ubtraction

Remember... multiplications and divisions before additions and subtractions — unless **parentheses** tell you otherwise.

More generally, the order you should perform operations is as follows:

Order of Operations:

() []	**1.** Evaluate anything inside parentheses (or brackets)
y^2	**2.** Evaluate exponents
× ÷	**3.** Multiply and divide from left to right
+ –	**4.** Add and subtract from left to right

Example 4

Evaluate: $3^2 \times 5 - (20 - 2) \div 3^2$.

Solution

$3^2 \times 5 - (20 - 2) \div 3^2 = 3^2 \times 5 - \mathbf{18} \div 3^2$ — Parentheses first
$= \mathbf{9} \times 5 - 18 \div \mathbf{9}$ — Then exponents
$= \mathbf{45} - 18 \div 9$ — × and ÷, from left to right
$= 45 - \mathbf{2}$
$= \mathbf{43}$ — + and –, from left to right

Example 5

Alan and Jessica are each trying to calculate the expression $24 \div 3 - 2^2$. Their work is shown below. Who has the correct answer?

Alan: $24 \div 3 - 2^2 = \mathbf{8} - 2^2$
$= \mathbf{6^2}$
$= \mathbf{36}$

Jessica: $24 \div 3 - 2^2 = 24 \div 3 - \mathbf{4}$
$= \mathbf{8} - 4$
$= \mathbf{4}$

Solution

Alan performed the division, then the subtraction, then evaluated the exponent. That is **incorrect** — he should have found the **exponent first**.

Jessica evaluated the exponent first, then performed the division, then the subtraction. This is the right order, so **Jessica has the right answer**.

Check it out:

Use the same order of operations inside parentheses as well.
For example: $10 - (4^2 + 2 \times 3)$
You need to evaluate the parentheses first, and you follow the normal PEMDAS rules to do that.
So evaluate $4^2 + 2 \times 3$ in this order: exponent, multiplication, addition.
Therefore $10 - (4^2 + 2 \times 3)$
$= 10 - (\mathbf{16} + 2 \times 3)$
$= 10 - (16 + \mathbf{6})$
$= 10 - \mathbf{22}$
$= \mathbf{-12}$

Guided Practice

Evaluate the expressions shown in Exercises 9–18.

9. $8 + (10 - 2) \div 4$
10. $4 - 2 \times 16$
11. $5 \times 7 + 7 \times 5$
12. $7 \times 3 - 8 \div 2 + 6$
13. $3^2 + 4 \times 9$
14. $2^3 - 9 \div 3$
15. $(4 + 1)^2 - 12 \div 4 - 1$
16. $3 \times (4^2 - 5) + 11$
17. $(6^2 - 3^2) \div (6 - 3)^2$
18. $(15 \div 3 - 2) \times (3^3 - 5 \times 4)$

Order Rules Also Apply to Expressions with Variables

The order rules apply to **all types** of expressions, including those with **variables**.

Example 6

Evaluate $x + yz$ when $x = -3$, $y = 9$, and $z = 10$.

Solution

Substitute in your values for x, y, and z	$\mathbf{-3 + 9 \times 10}$
There are no parentheses or exponents	$-3 + 9 \times 10$
Carry out the multiplication	$-3 + \mathbf{90}$
Carry out the addition, giving the answer	$\mathbf{87}$

Example 7

Evaluate $19 - f^2 \times (w + q)$ when $f = 3$, $w = 5$, and $q = 2$.

Solution

Substitute in your values for f, w, and q	$\mathbf{19 - 3^2 \times (5 + 2)}$
Evaluate the parentheses	$19 - 3^2 \times \mathbf{7}$
Evaluate the exponents	$19 - \mathbf{9} \times 7$
Carry out the multiplication	$19 - \mathbf{63}$
Carry out the subtraction, giving the answer	$\mathbf{-44}$

Guided Practice

Evaluate the expressions shown in Exercises 19–21, given that $f = 2$, $j = 3$, and $g = 19$.

19. $g - j \times 8$
20. $(f + g) \times j + 4$
21. $(j^2 - f^2) \div 5$

Evaluate the expressions shown in Exercises 22–24, given that $s = 10$, $k = 3$, and $q = 1$.

22. $s \times k + qs + 23$
23. $(k^2 - 1) \div 4q$
24. $q - sk^3$

Independent Practice

Evaluate Exercises 1–4, given that $w = 2$, $b = 8$, $c = -0.5$.

1. $b \times (w - c)$

2. $w \times b + w \times c$

3. $(6 - w)^2$

4. $(b + w + 2 \times c) \div (b - 2 \times c)$

5. Felipe and Sylvia are trying to evaluate the expression $2 \times 18 - 6$.

Felipe: $2 \times 18 - 6$
$= \mathbf{36} - 6$
$= \mathbf{30}$

Sylvia: $2 \times 18 - 6$
$= 2 \times \mathbf{12}$
$= \mathbf{24}$

Who is correct? What has the other person done wrong?

Don't forget:
If there are no parentheses, evaluate multiplications and divisions from left to right.

Which of the expressions in Exercises 6–11 are true, and which are false? For those that are false, explain what is wrong.

6. $(2 + 3) \times 6 = 2 + 3 \times 6$

7. $(7 + 5) - 3 = 7 + 5 - 3$

8. $(100 \div 2) \div 25 = 100 \div 2 \div 25$

9. $10 \div 2 \times 5 = 10 \div (2 \times 5)$

10. $3 + (6 \times 9) = 3 + 6 \times 9$

11. $6 \times 5^2 = (6 \times 5)^2$

12. Carly wants to exercise for a total of 50 hours this month. She exercised 1.5 hours for each of the first eight days. Then, she exercised 2 hours for each of the next four days. Write and evaluate an expression for the remaining number of hours she has to exercise.

13. Mr. Chang earns \$12 per hour baking bread at the bakery. He worked 7 hours on Thursday and c hours on Friday. Write an expression of the form $a \times (b + c)$ that can be used to calculate how much Mr. Chang earned on Thursday and Friday.

14. Given that Mr. Chang worked for 8.5 hours on Friday, find the value of the expression you wrote for Exercise 13.

Now try these:
Lesson 2.1.4 additional questions — p432

Which of the expressions in Exercises 15–19 are true, and which are false? For those that are false, explain what is wrong.

15. $h - r + q = h - (r + q)$

16. $a \times (5 + g) = a \times 5 + g$

17. $16 \times f \div 3 + 2 = 16 \times f \div (3 + 2)$

18. $5 + j - w = (5 + j) - w$

19. $18y - 2j = (18 - 2) \times (y - j)$

Round Up

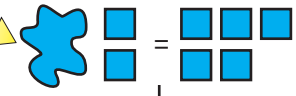
Hopefully you can now see how important the order of operations is when evaluating expressions. You'll need to be able to get the order right from now on — so make sure you memorize it.

Section 2.2 introduction — an exploration into:

Equations with Algebra Tiles

You can use red and blue tiles to represent math equations. Equations show that two things are "balanced," and you're not allowed to do anything that might make them unbalanced. This means you can add or remove tiles, as long as you do the same to both sides.

You can use tiles to represent equations. These tiles show that $x + 2 = 5$.



When you have an equation, you can add or remove tiles — as long as you add or remove exactly the same tiles on **both sides**.

If you can get x by itself, you can find its value. Here, $x = 3$.

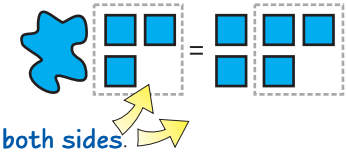


Example

Show the equation $x + 3 = 5$ with algebra tiles, and then find the value of x.

Solution

The equation $x + 3 = 5$ looks like the picture on the right.

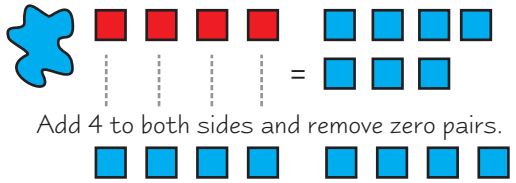


Remove 3 squares from **both sides** to get x by itself.
This tells you that $x = 2$.

For some equations, you need to use **zero pairs** before you can get x by itself.

Example

Show $x - 4 = 7$ with tiles, and find the value of x.



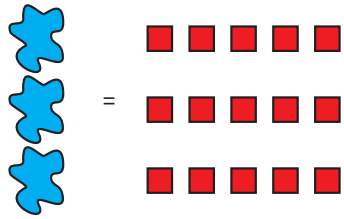
Solution

The equation $x - 4 = 7$ looks like this.

Add 4 tiles to **both sides**, and then **remove zero pairs**.
This leaves x on one side of the equation, and 11 blue tiles on the other side. So $x = 11$.

Example

Show $3x = -15$ with tiles, and find the value of x.



Solution

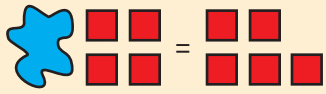
$3x = -15$ looks like the picture on the right. Put the x's in a line, and then put the same number of tiles next to every x.
For every x, there are 5 red tiles. This tells you that $x = -5$.

Exercises

1. Use algebra tiles to model each of these equations, and then find the value of x.

a. $x + 6 = 9$ **b.** $x - 3 = 8$ **c.** $x + 2 = -5$ **d.** $2x = 10$ **e.** $3x = -12$

2. Write the equation that is represented by these tiles. Then solve the equation to find the value of x.



Round Up

Keeping equations balanced by doing the same to both sides is one of the most important things you'll ever learn in math. There will be a lot more about this in the next Section.

Lesson 2.2.1

Equations

You've done plenty of work with expressions — now it's time to tackle equations. They're both quite similar — both involve variables, numbers, and combinations of the usual +, –, ×, and ÷ operations.

California Standard:

Algebra and Functions 1.1

Write and solve one-step linear equations in one variable.

What it means for you:

You'll learn about equations and how they're related to expressions.

Key words:

- equation
- guess and check
- solve

An Equation Tells You That Two Things are Equal

An equation tells you that **something is equal to something else**.
These are all examples of equations: $14 + 6 = 20$
$9 - 3 = 4 + 2$
$4y + x = x - 7$

Writing an equation is similar to writing an **expression**. It may involve a **real-life** problem, and you may have to use **variables** to represent unknown values.

Example 1

Richard spent \$3.50 on fruit and \$7.26 on vegetables.
He then spent \$$x$ on meat, making a total of exactly \$20.
Write an equation to represent this.

Solution

Richard spent $3.50 + 7.26 + x$ dollars, which is equal to \$20.
Writing this as an equation, $\mathbf{3.50 + 7.26 + x = 20}$.
This can be simplified to $\mathbf{10.76 + x = 20}$.

Example 2

Edward and Bianca each wrote a story in English class. Bianca's story has 31 words more than Edward's. Write an equation to represent this.

Solution

Create variables — call the number of words in Bianca's story b and the number of words in Edward's story e.

Although both b and e are unknown amounts, the question says that ***b* is 31 greater than *e***.

So, a possible equation is $\mathbf{e + 31 = b}$.
You could also have written $\mathbf{b - 31 = e}$.
Both equations show that b is 31 greater than e.

Check it out:

An equation involving variables can be true for all values of the variable — for example, $y + y = 2y$ (this kind of equation is usually called an identity).
Or it can be true for only particular values of the variable — for example, $2y + 3 = 11$, which is true only if $y = 4$.
Finding the values that make an equation true is called solving the equation, and there's more about this later. (The equations in Examples 1 and 2 are only true for particular values.)

Guided Practice

Write equations to represent the situations described in Exercises 1–6.

1. Jose scored 10 percent more than William on a class test.
2. Theresa owns two pairs of shoes more than Kimberly.

3. Richard and Tim each bake cakes. Richard notices that Tim's cake is three times as heavy as his.
4. Isaac goes to the county fair. Because he has been saving up, he has seven times as much money as his sister to spend on rides.
5. Laura's dad runs a newsstand that sells bottles of water. Laura knows that five small bottles of water contain the same amount as two large bottles.
6. Monica has a candy bar that is a third the size of Julio's.

Equations Can Be Used to Find the Value of Variables

Equations like $y + 6 = 16$ tell you enough information to find out y.

One way to find y is to **guess** at it again and again until you find the right answer. Sometimes this is called the "**guess and check**" method.

Check it out:
Using an equation to find the value of an unknown variable is sometimes called solving the equation.

Example 3

Find the value of y if $y + 6 = 16$.

Solution

First you need to guess a value for y. Then you can use it in the equation and check if it makes the equation true.

Try $y = 5$: $5 + 6 = 16$ is **not true**.
In fact, $5 + 6 = 11$, which is **too small**.

Try $y = 12$: $12 + 6 = 16$ is also **not true**.
In fact, $12 + 6 = 18$, which is **too big**.

Try $y = 10$: $10 + 6 = 16$, which **is true**. So y **does** equal **10**.
This means the solution to the equation is $y = 10$.

Check it out:
As shown in Example 4, one way of doing "guess and check" is thinking of the equation as a question. Once you can answer "yes" to the question "$7h = 56$?", you've got the right value of h.

Example 4

Julia uses the expression $7h$ to find out how many dollars she earned doing yard work. h represents the number of hours she worked. If Julia earned \$56, then how many hours did she work?

Solution

An equation for the situation is $7h = 56$.
Try guessing different values of h, then see if they're right.

h	$7h$	$7h = 56$?
7	49	No
10	70	No
8	**56**	**Yes**

If Julia worked for **7 hours**, she would have earned only \$49, which is **too little**.

If Julia worked for **10 hours**, she would have earned \$70, which is **too much**.

Julia must have worked for **8 hours** — because when $h = 8$, the equation $7h = 56$ is **true**.

Guided Practice

Use "guess and check" to solve the equations in Exercises 7–10.

7. $x + 14 = 30$ **8.** $2z + 16 = 38$

9. $9a - 12 = 69$ **10.** $3z \div 7 = 6$

11. Luis uses the expression $\$3 \times m$ to work out how much money he spends calling his relatives in Europe for m minutes. His last phone call cost \$60. Write an equation to represent this situation and then find out how long he spoke for.

12. Pedro works out how much he must pay toward a food bill by using the formula $d - \$3.24$, where d is the cost of the entire bill. He ends up paying \$2.97. Write an equation for this situation and then find out how much the food bill must have been.

13. What value of y satisfies the equation $111 = 19y + 16$?

Independent Practice

Now try these:
Lesson 2.2.1 additional questions — p433

Use "guess and check" to solve the equations in Exercises 1–4.

1. $a + 20 = 57$ **2.** $3b + 9 = 15$

3. $7c - 22 = 20$ **4.** $12d \div 8 = 6$

5. Brandon has $\$t$ in his savings jar. He puts in an extra \$12. If the total amount in the jar is n, write an equation linking n and t.

6. Solve the equation you wrote for Exercise 5 when $n = 48$.

7. Cody is playing an old video game. In it you get 400 points every time you collect a ring and 800 points every time you collect an egg. Write an equation about p, the total number of points you get for collecting r rings and e eggs.

8. Guadalupe has started going to dance class. Each class costs \$3 and she spent \$23.50 on dance shoes. Including the shoes, she has spent \$71.50 in total. Write an equation for this and solve it to find out how many lessons Guadalupe has been to.

9. It takes 2 eggs to make 20 cookies, and 4 eggs to make 40 cookies. Which equation best describes the number of eggs, x, that it takes to make y cookies?

$y = 10x$ $x = 10y$ $y + x = 10$ $y - x = 10$

Round Up

Equations look like expressions, but they contain a "=" — which tells you that something is equal to something else. Don't worry if you're not 100% comfortable with equations yet — all of the Lessons in this Section are about equations, so you'll have many chances to practice.

Lesson 2.2.2

Manipulating Equations

California Standard:

Algebra and Functions 1.1

Write and solve one-step linear equations in one variable.

What it means for you:

You'll learn about how to manipulate linear equations, which you'll use in Lessons 2.2.3 and 2.2.4 when you're solving them.

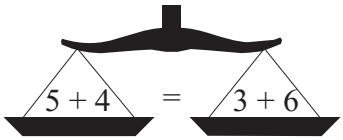
Key words:

- equation
- manipulate

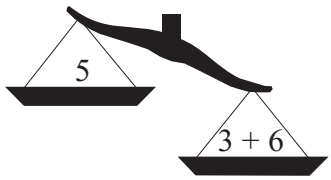
Now that you've met equations, it's time to look at the things you can do to change an equation without making it untrue.

You Have to Keep Equations Balanced

An equation is like a scale. The bit before the equals sign **has the same value** as the bit after the equals sign, so the scale is balanced.

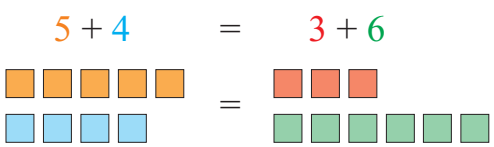


When manipulating equations, you have to **keep the scale balanced**. You can't take 4 from one side and not from the other because then the two sides aren't equal.



The only way to keep the scale balanced is to **always do the same thing to both sides**.

Here's another way to show the same thing:



There are a total of **9 boxes** on each side of the equals sign — so the equation is **balanced**.

If you **take** any **3 boxes** from the **left-hand side**, you also need to take **3 boxes** from the **right-hand side** to keep the equation balanced.



Both sides still have the same total number of colored boxes.

You still have to keep both sides of an equation **balanced** when there are **variables** involved.

Example 1

The variables x and y are linked by the equation $x + y = 10$.
Show how to keep the equation balanced if:
(i) 3 is added to the left-hand side,
(ii) y is subtracted from the right-hand side.

Solution

(i) If 3 is added to the left-hand side, then you also have to add 3 to the right-hand side. This gives:
$x + y + 3 = 10 + 3$, which can be simplified to $x + y + 3 = 13$.

(ii) If y is subtracted from the right-hand side, then you also have to subtract y from the left-hand side. This gives:
$x + y - y = 10 - y$, which can be simplified to $x = 10 - y$.

Check it out:

Notice how $x + y - y$ can be simplified to just x.
The "$+ y - y$" part cancels out.

Check it out:
Notice how the "+ 7 – 7" part cancels out.

Check it out:
Notice how you've found the value of x by subtracting something from both sides of the equation.

Example 2

Subtract 7 from both sides of the equation $x + 7 = 23$.
What does the result tell you about the value of x?

Solution

$x + 7 = 23$
So $x + 7 - 7 = 23 - 7$
So $\mathbf{x = 16}$.

Example 3

Felipe has the equation $j + 26 = 32$.

He subtracts 26 from the left-hand side to get $j = 32$.
Is that the right value for j?

Solution

Felipe has changed one side of the equation without changing the other in the same way. It is no longer balanced.

This value for j won't make the equation true.

He should have done the same to both sides.

$j + 26 = 32$
$j + 26 - 26 = 32 - 26$ **Subtract 26 from both sides**
$j = 6$ **Then simplify to get j on its own**

Guided Practice

Fill in the missing parts of Exercises 1–6.

1. $y = 3$
$y + 5 = ?$

2. $w = 23$
$w \times 6 = ?$

3. $a = 27$
$a \div 9 = ?$

4. $b = 7$
$b + 9 = ?$

5. $x = 15$
$x \div 5 = ?$

6. $h = 12$
$h \times 4 = ?$

Equations can involve more than one variable. As always, keep both sides of the equation **balanced**, by doing the same to both sides of the equation.

Example 4

Adriana buys 3 wrenches and 2 screwdrivers. Each wrench costs \$$w$ and each screwdriver \$$s$. She spends \$15.66, so she writes $3w + 2s = 15.66$.

She decides to work out the cost of three times as many wrenches and screwdrivers by multiplying the left-hand side of the equation by 3.

What must she do to work out the right-hand side?

Solution

She needs to multiply the right-hand side by 3.
$3 \times (3w + 2s) = 3 \times 15.66$, or $\mathbf{3 \times (3w + 2s) = 46.98}$

Check it out:
Notice that \$46.98 is the cost of 3 lots of "3 wrenches and 2 screwdrivers" — which is the same as 9 wrenches and 6 screwdrivers.

Check it out:

When you're manipulating equations, write the different stages down the page, lining them up by their = signs. That makes it easier to see what operations have been carried out on each side.

Guided Practice

Fill in the missing parts of Exercises 7–12.

7. $u = f + 2$
$u - 5 = ?$

8. $j = z + 26$
$j - 26 = ?$

9. $r + j = 9$
$(r + j) \times 15 = ?$

10. $6 = q$
$6 - 19 = ?$

11. $f = b$
$fz = ?$

12. $? = m + 3$
$2 = m$

Explain what step of the work is incorrect in Exercises 13–18.

13. $f = z + 19$
$f = z$

14. $t = 3k$
$t - 3 = k$

15. $6 + g = 7$
$5 + g = 8$

16. $p = 7$
$2p = 27$

17. $h = 11 + j$
$h - 11 = j - 11$

18. $15q = 9$
$q = 9 \times 15$

Now try these:

Lesson 2.2.2 additional questions — p433

Independent Practice

1. Joshua and Ashley both try manipulating the same equation. Their work is shown below. Have either of them made a mistake? If so, what have they done wrong?

Joshua	Ashley
$9w = 27$	$9w = 27$
$9w + 16 = 27 \div 16$	$9w + 16 = 43$

2. Vanessa notices that the cost of a big bag of oranges, b, is exactly \$0.45 less than the cost of a sandwich, s. Write an equation involving subtraction to represent this.
3. Add 0.45 to both sides of your equation from Exercise 2.
4. Using your answer for Exercise 3, if a big bag of oranges costs \$2.25, then what does a sandwich cost?

Fill in the missing parts of Exercises 5–10.

5. $w + p = 22$
$w + p + 9 = ?$

6. $16 - h = 5$
$? = 5 + h$

7. $9h = 9$
$9h \div 9 = ?$

8. $h \div 3 = 4$
$3 \times h \div 3 = ?$

9. $w + j = 4$
$? = 4 + w + j$

10. $m \times 2 = 4$
$m = ?$

Round Up

The concepts here are really important — in the next Lesson, you'll manipulate equations to solve them. Remember — you must do exactly the same thing to both sides of the equation. Always.

Lesson 2.2.3

Solving + and – Equations

California Standard:

Algebra and Functions 1.1

Write and solve one-step linear equations in one variable.

What it means for you:

You'll manipulate + and – equations to solve them without using "guess and check."

Key words:

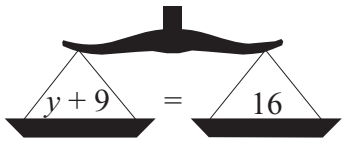
- equation
- solve

As you might have guessed, all that manipulating equations was leading up to something, and this is it. This Lesson is about a fast way of solving equations by manipulating sides.

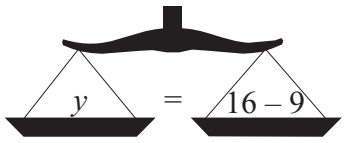
Manipulating Is Faster Than Guess and Check

You can use an equation to **find out the value of a variable**. You've done this already using "guess and check," but a quicker way is to **manipulate the equation**.

An equation like $y + 9 = 16$ is balanced just like one with only numbers. To find the value of y, you need to get the variable **alone on one side of the equals sign**.



If the variable has something **added** to it, **use subtraction** to get it on its own. In $y + 9 = 16$, subtract 9 from both sides to get y on its own.



You can do exactly the same thing without drawing the scales.

$$\begin{aligned} y + 9 &= 16 \\ y + 9 - 9 &= 16 - 9 \\ y &= 7 \end{aligned}$$

The "+ 9" and "– 9" cancel each other out.

You can **check** that $y = 7$ is the correct solution by substituting it back into the original equation:

$7 + 9 = 16$ — this is true, so $y = 7$ is correct.

Check it out:

The "+ 19 – 19" is crossed out because the values cancel. It's okay to cross things out — but do it lightly, so that all your work can still be seen.

Example 1

Alicia and Paula are each solving the equation $x + 19 = 37$. Their work is shown below. Who solved the problem correctly?

Alicia

$$\begin{aligned} x + 19 &= 37 \\ x + \cancel{19} - \cancel{19} &= 37 \\ x &= 37 \end{aligned}$$

Paula

$$\begin{aligned} x + 19 &= 37 \\ x + \cancel{19} - \cancel{19} &= 37 - 19 \\ x &= 18 \end{aligned}$$

Solution

Paula is correct. She has remembered to subtract 19 from both sides of the equation. Alicia's solution doesn't make sense — she's subtracted 19 from one side of the equation, but not the other.

Guided Practice

Solve the equations given in Exercises 1–14.

1. $x + 11 = 17$
2. $x + 10 = 20$
3. $k + 75 = 200$
4. $14 + x = 14$
5. $y + 191 = 87$
6. $w + 8 = -1$
7. $3.8 + y = 19.1$
8. $33.4 + z = 47.2$
9. $1.3 = y + 0.5$
10. $q + 18 = 20.2$
11. $g + (-8) = 13$
12. $-9 = f + 9$
13. $v + 23 = -10$
14. $i + (-9) = 10$

15. Yesenia is given two new video games for her birthday. She now has 18. Write an equation for this, using v for the number of video games she had before her birthday. Solve it to find v.

16. Alejandro puts $14.50 into his bank account. He now has a total of $34.15. Write an equation for this, using m for the amount he had in the bank before the deposit. Then solve it to find m.

Check it out:
Don't let decimals or negative numbers put you off. Just do exactly the same as you have been doing and you'll still get the right answer.

Subtraction Equations Can Be Solved with Addition

If the variable has something **subtracted** from it, then you need to **add to both sides**. This way, you can get the **variable on its own**.

Taking something away and then giving it back is the same as not doing anything at all.

$$q - 5 = -3$$
$$q - \cancel{5} + \cancel{5} = -3 + 5 \qquad \text{Add 5 to both sides}$$
$$q = 2 \qquad \text{Then simplify}$$

Example 2

George and Kayla are each solving the equation $u - 23 = 19$. Their work is shown below. Who solved the problem correctly?

George

$$u - 23 = 19$$
$$u - \cancel{23} + \cancel{23} = 19 + 23$$
$$u = 42$$

Kayla

$$u - 23 = 19$$
$$u - \cancel{23} + \cancel{23} = 19$$
$$u = 19$$

Solution

George is correct. He has remembered to add 23 to both sides of the equation. Kayla's solution doesn't make sense — she has added 23 to one side of the equation, but not the other.

Example 3

Solve the equation $x - (-9) = -14.3$.

Solution

$$x - (-9) = -14.3$$
$$x - \cancel{(-9) + (-9)} = -14.3 + (-9)$$
$$x = -14.3 + (-9)$$
$$x = -23.3$$

Don't forget:
There's more about solving this kind of equation in Lesson 2.2.5.

Guided Practice

Solve the equations given in Exercises 17–30.

17. $q - 5 = 12$
18. $q - 6 = -3$
19. $x - 3 = 0$
20. $w - 3 = 6$
21. $n - 2 = 13$
22. $x - 12 = 14$
23. $12 = h - 6$
24. $j - 6 = 5$
25. $19 = r - 15$
26. $b - 11 = 2.7$
27. $g - 0.1 = 0.9$
28. $y - (-2) = 8$
29. $e - 0.3 = 1.9$
30. $9 = v - 0$

31. Ted is playing a game of snap. So far he has put down 5 cards, leaving him holding 7. Write an equation for this, using n for the number of cards Ted had at the start of the game. Solve it to get n.

Independent Practice

Solve the equations given in Exercises 1–8.

1. $14 + k = 15$
2. $11 = t + 3$
3. $j + 1 = 3.9$
4. $15 = p + 16$
5. $c - 3 = 0$
6. $1.5 = k - 1.5$
7. $f - 0.6 = 10.2$
8. $1.3 = a - 2.8$

In Exercises 9–16, write an equation for the situation described and then solve it. Use x to represent the variable.

9. When eight is subtracted from a number, the result is 25.
10. Three plus a number is 30.
11. Nine more than a number is twenty-three.
12. When a number is decreased by 8.9, the result is 41.4.
13. A number increased by 9.8 is equal to 17.
14. A number decreased by forty-two is equal to ten.
15. Twenty is the sum of a number and three.
16. When 5.6 is subtracted from a number, the result is 17.2.

17. Mr. Jenkins lost 14 pounds on a diet. He now weighs 210 pounds. Write a subtraction equation for this situation, using w to represent his weight before the diet. Then solve the equation to find w.

18. Monique's hair has grown by an inch and a half since it was last cut and it is now 11 inches long. Write an addition equation to represent this situation. Use h to represent the length of hair right after the last haircut. Then solve the equation to find h.

19. Karen is 61 inches tall, which is 8 inches shorter than her mother. Which equation can be used to represent her mother's height, h?

$h + 61 = 8$ $h = 61 - 8$ $h - 8 = 61$ $h = 8 \times 61$

Now try these:
Lesson 2.2.3 additional questions — p433

Round Up

In the world of equation manipulation, "+" gets rid of "–", and "–" gets rid of "+." There's really not much more to it than that, as long as you play by the rules — you must always do exactly the same thing to both sides.

Lesson 2.2.4

Solving × and ÷ Equations

California Standard:

Algebra and Functions 1.1

Write and solve one-step linear equations in one variable.

What it means for you:

You'll do more equations, this time dealing with equations containing multiplications and divisions.

Key words:

- equation
- solve

Don't forget:

$5t \div 5 = t$

More equations, but this time with multiplication and division in them. The same rules as last Lesson apply — you need to get the variable alone on one side of the equals sign.

Divide to Cancel Out a Multiplication

To solve equations like $5t = -20$, you still need to get the variable, t, on its own. The variable has been **multiplied** by a number, 5 — so you can **get the variable on its own by dividing**. In this case, you need to divide by 5.

$$\begin{aligned} 5t &= -20 \\ 5t \div 5 &= -20 \div 5 \\ t &= -20 \div 5 \\ t &= -4 \end{aligned}$$

divide both sides by the same thing the variable is multiplied by

Example 1

Solve the equation $7r = 42$.

Solution

$$\begin{aligned} 7r &= 42 \\ 7r \div 7 &= 42 \div 7 \\ r &= 42 \div 7 \\ r &= 6 \end{aligned}$$

Divide both sides by 7, then simplify to work out r.

Exactly the same rule works if the variable has been multiplied by a **negative number**.

Example 2

Solve the equation $-6q = 54$.

Solution

$-6q$ means $-6 \times q$, so you need to divide by -6.

$$\begin{aligned} -6q &= 54 \\ -6q \div (-6) &= 54 \div (-6) \\ q &= 54 \div (-6) \\ q &= -9 \end{aligned}$$

Divide both sides by –6, then simplify to work out q.

Don't forget:

"•" just means "multiply." The phrase "5 multiplied by k" can be written as "$5 \bullet k$" or "$5 \times k$" or just "$5k$."

Guided Practice

Solve the equations given in Exercises 1–8.

1. $3t = 27$
2. $16w = 48$
3. $7x = 28$
4. $r \times 6 = 7.2$
5. $12y = 13.2$
6. $-9p = -72$
7. $-5 \bullet k = 30$
8. $h \times 8 = -56$

Multiply to Cancel Out a Division

Division and multiplication cancel each other out, so you can **use multiplication to solve division equations**.

Don't forget:
Division and multiplication are "opposites" — see Lesson 1.2.2 to remind yourself why.

$$\begin{aligned} m \div 11 &= 6 \\ m \div 11 \times 11 &= 6 \times 11 \\ m &= 6 \times 11 \\ m &= 66 \end{aligned}$$

multiply both sides by the same thing the variable is divided by

Example 3

Solve the equation $q \div 5 = 7$.

Solution

$$\begin{aligned} q \div 5 &= 7 \\ q \div 5 \times 5 &= 7 \times 5 \\ q &= 7 \times 5 \\ q &= 35 \end{aligned}$$

Multiply both sides by 5 then simplify to work out q.

Example 4

Sienna and Raoul are each solving the equation $p \div 6 = 12$.
Their work is shown below. Who solved the problem correctly?

Raoul

$$\begin{aligned} p \div 6 &= 12 \\ p \div 6 \times 6 &= 12 + 6 \\ p &= 18 \end{aligned}$$

Sienna

$$\begin{aligned} p \div 6 &= 12 \\ p \div 6 \times 6 &= 12 \times 6 \\ p &= 72 \end{aligned}$$

Solution

Sienna is correct. Raoul has not done the same to both sides of the equation. He has multiplied the left-hand side by 6, but added 6 to the right-hand side.

Guided Practice

Solve the equations given in Exercises 9–20.

9. $t \div 2 = 3$
10. $w \div 5 = 11$
11. $q \div 4 = 13$
12. $u \div 8 = 5$
13. $x \div 1.5 = 3$
14. $p \div 5 = 2.7$
15. $g \div -3 = 9$
16. $s \div -7 = 6$
17. $d \div -6 = -12$
18. $j \div 13 = -15$
19. $y \div 0.9 = 9.1$
20. $e \div 1.2 = -2.3$

21. A builder wants to buy a pipe that is long enough to cut into four 30-inch pieces, as shown below. Write and solve a division equation that can be used to find the length of the pipe, p, that the builder should buy.



Independent Practice

Now try these:
Lesson 2.2.4 additional questions — p434

Solve the equations given in Exercises 1–8.

1. $7x = 28$
2. $0.5s = 10$
3. $w \div 8 = 1$
4. $34p = 0$
5. $5 \bullet r = -10$
6. $t \div 0.5 = 10$
7. $7 = z \div -8$
8. $-7.2 = -6y$

9. Juan likes to swim laps at the local pool. The pool has a length of 25 meters. Juan wants to swim 350 meters. Write and solve a multiplication equation to find out how many lengths Juan needs to swim.

10. Taylor earned $94.50 last week at her job, where she earns $6.75 an hour. Solve the equation $94.50 = 6.75h$ to find out how many hours, h, she worked.

Don't forget:
To check that an answer is correct, put it back into your original equation. If you put 9 back into the equation $3x = 12$, you can see right away that it doesn't make sense.

11. When Ross solved the equation $12 = 3x$, he made a mistake. His work is shown below. What mistake did Ross make?

$$3x = 12$$
$$3x - 3 = 12 - 3$$
$$x = 9$$

12. A poster has an area of 800 square inches, and a length of 40 inches. Write and solve a multiplication equation that can be used to find the width.

13. A farmer has a rectangular plot of land that he divides into four smaller plots, each with an area of 5000 square yards. Write and solve a division equation to find the area of the original plot.

Solve the equations given in Exercises 14–21. These will give you practice on both "+" and "–" questions and "×" and "÷" questions.

14. $f + 9 = 3$
15. $12k = 288$
16. $-7 = n \div 4$
17. $b - 23 = 2.1$
18. $q - 0.23 = 1.66$
19. $-2d = 8$
20. $4.2 = m \div 3$
21. $y \div 5 = -80$

In Exercises 22–27, write an equation for the situation described and then solve it. Use x to represent the variable.

22. A number divided by seven is equal to fifteen.
23. The product of a number and three is equal to negative nine.
24. A number multiplied by 2 is equal to 8.4.
25. A number added to 9 is 37.
26. A number divided by negative three is equal to five.
27. Negative fifteen taken away from a number is 6.2.

Round Up

By now you should be used to always doing the same thing to both sides of an equation. The main thing from this Lesson is that you can undo multiplication with division and you can undo division with multiplication. Just keep practicing...

Lesson 2.2.5

Graphing Equations

Solving equations is one of the most important things you do in math. That means you need plenty of practice at it. And you need to understand exactly why this method works. This Lesson starts with a summary of things you've seen in the previous Lessons in this Section.

California Standard:

Algebra and Functions 1.1

Write and solve one-step linear equations in one variable.

What it means for you:

You'll see how what you've learned about equations matches up to the number line, and you'll practice deciding what sort of equation you're dealing with.

Key words:

- equation
- graph
- number line

There Are Different Types of One-Step Equations

A **one-step equation** is one that can be solved in one step by either adding, subtracting, multiplying by, or dividing by **one thing**.

There are four main types. For example:

(i) $a + 3 = 4.2$ — solve by **subtracting** 3 from both sides to get $a = 1.2$
(ii) $s - 7 = 12$ — solve by **adding** 7 to both sides to get $s = 19$
(iii) $9m = 27$ — solve by **dividing** both sides by 9 to get $m = 3$
(iv) $d \div 8 = 2$ — solve by **multiplying** both sides by 8 to get $d = 16$

Before you can solve an equation, you must be able to spot what kind of equation you have.

Example 1

Explain how you would solve each of the following equations.
(i) $x + 12 = 4.2$, (ii) $-7y = 14$, (iii) $a - 87 = 1$

Solution

(i) $x + 12 = 4.2$ Solve by subtracting 12 from both sides: $x = -7.8$

(ii) $-7y = 14$ Solve by dividing both sides by -7: $y = -2$

(iii) $a - 87 = 1$ Solve by adding 87 to both sides: $a = 88$

Don't forget:

$-7y = 14$ means $-7 \times y$, so it's a multiplication equation that you solve by dividing.
(It is not the same as $y - 7$, or $7 - y$.)

Sometimes you can look at an equation in more than one way.

Example 2

Solve $n + (-2) = 6.3$.

Solution

There are two ways to think about $n + (-2) = 6.3$.

(i) You can say that "***n* has –2 added to it, so solve the equation by subtracting –2 from both sides**."
This gives $n + (-2) - (-2) = 6.3 - (-2)$.
But $6.3 - (-2)$ is the same as $6.3 + 2$, so this gives $n = 8.3$

(ii) Or you can rewrite the equation **before** solving it, using the fact that adding a negative is the same as subtracting a positive: $n - 2 = 6.3$.
You can solve this by **adding 2 to both sides**.
This gives $n - 2 + 2 = 6.3 + 2$, which gives $n = 8.3$

Whichever method you use, it all comes down to the same thing.

Don't forget:

Adding a negative number is the same as subtracting a positive number.
And subtracting a negative number is the same as adding a positive number.

Don't forget:

Notice how closely addition and subtraction are linked. Addition problems can be turned into subtraction problems. Similarly, subtraction problems can be turned into addition problems.

Equations with **division** come in two forms. You may see the ÷ written in or you may see the equation written as though it is a **fraction**.

Example 3

Solve $\frac{h}{-9} = 7$.

Solution

This is the same as $h \div (-9) = 7$.
Solve by multiplying by –9.

$h \div (-9) \times (-9) = 7 \times (-9)$, which gives $\boldsymbol{h = 7 \times (-9) = -63}$.

Guided Practice

Identify the type, and solve the equations in Exercises 1–12.

1. $3t = 27$ **2.** $-4 + q = 11$ **3.** $g \times 2.4 = 4.8$

4. $-8 + z = 8$ **5.** $d \times 19 = 95$ **6.** $-4j = 20$

7. $\frac{u}{2.9} = 3$ **8.** $b - (-3) = -9$ **9.** $12 + j = 9$

10. $n \div 7 = 4$ **11.** $k + (-3) = 14$ **12.** $\frac{p}{12} = 8.2$

Show Your Solutions on the Number Line

Sometimes in math, you need to draw your solutions on a **graph**.
For the equations in this Lesson, that means drawing a **number line**.

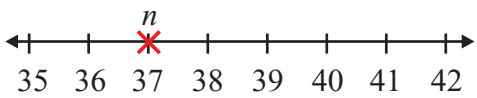
Example 4

Solve the equation $n + 5 = 42$ and graph your result.

Solution

Solve the equation in the normal way. Here, you need to **subtract 5 from both sides** to get $\boldsymbol{n = 37}$.

Now graph this solution on a number line.



Don't forget:
There's a lot more information about the number line in Chapter 1.

But notice also that a number line can help you **solve** an equation.

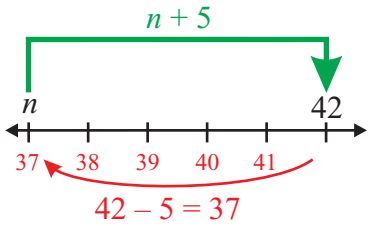
Example 5

Solve the equation $n + 5 = 42$.

Solution

The equation $n + 5 = 42$ tells you that there is a position on the number line, n, and that 42 is 5 places **to the right** of it.

That means that n is 5 places **to the left** of 42. That can be written as $n = 42 - 5$. So $\boldsymbol{n = 37}$.



Check it out:
This isn't a new way to solve an equation — it's just a different way of thinking about the method you've already seen.
Solving $n + 5 = 42$ by subtracting 5 from both sides is equivalent to moving 5 places to the left on a number line.

Example 6

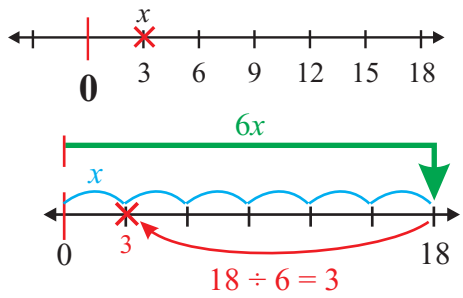
Solve the equation $6x = 18$ and graph your result.

Solution

You can divide both sides by 6 to find the solution $x = 3$.

Graphing the result means you have to draw a number line.

If you like, you can even use the number line to **solve** the equation. The equation $6x = 18$ tells you that 18 is 6 times further away from 0 on the number line than x.



Guided Practice

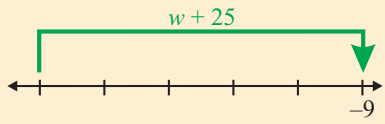
Solve the equations in Exercises 13–18, and graph your results.

13. $m + 10 = 9$ **14.** $t - 3 = 2$ **15.** $4x = 16$

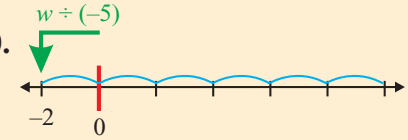
16. $r - 3 = -3$ **17.** $n + (-1) = 36.5$ **18.** $y \div 9 = 12$

Write an equation represented by each of the number lines in Exercises 19–20. Find the solution to each equation.

19.



20.



Independent Practice

Solve the equations in Exercises 1–12.
Show your results on a number line.

1. $d - 1 = 2$ **2.** $h \cdot 2.5 = 10$ **3.** $b + 62 = -189$

4. $11 = 12 + k$ **5.** $\frac{g}{-8} = 6$ **6.** $6 \times p = 174$

7. $8g = 64$ **8.** $v - 0.25 = 1.5$ **9.** $15 = 5k$

10. $r \div -12 = -3$ **11.** $j + 23 = 26$ **12.** $z - (-2) = 6$

13. During the week, Ramon drives t miles to school. On the weekend, he drives three times as far to a tennis club. The tennis club is 18 miles away. Write an equation for this and solve it to find out how far Ramon lives from school.

Now try these:
Lesson 2.2.5 additional questions — p434

Round Up

Equations become much easier to deal with if you practice — solving the equation just means working out how to get the variable alone. In the next Section, you'll be using expressions and equations to describe real-life situations.

Lesson 2.3.1

Section 2.3
Expressions About Length

California Standards:

Algebra and Functions 1.2
Write and evaluate an algebraic expression for a given situation, using up to three variables.

Algebra and Functions 1.3
Apply algebraic order of operations and the **commutative, associative**, and distributive **properties to evaluate expressions; and justify each step in the process.**

Algebra and Functions 3.1
Use variables in expressions describing geometric quantities (e.g., **$P = 2w + 2l$**, $A = \frac{1}{2}bh$, $C = \pi d$ — the formulas for **the perimeter of a rectangle**, the area of a triangle, and the circumference of a circle, respectively).

Algebra and Functions 3.2
Express in symbolic form simple relationships arising from geometry.

What it means for you:
You'll practice using expressions to represent the length and perimeter of objects.

Key words:
- length
- perimeter
- expression
- commutative
- associative

Don't forget:
Always check whether you need to include units in your answer — for example, inches, yards, miles...

You've seen lots of mathematical expressions — things like $x + 4$, or $y - 7$. You've also seen equations, and learned how to solve them — things like $x + 2 = 6$, for example.

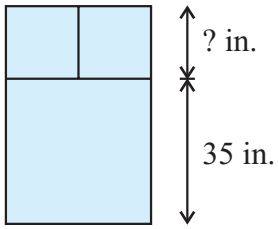
Now you need to apply what you know to other situations.

Expressions Can Describe Length

If you don't know the **length** of something, use a **variable** to represent it.

Example 1

Write an expression for the total height of the window on the right.



Solution

Choose a letter to represent the unknown height — w, say.

Then the total height of the window is **$(35 + w)$ inches**.

Pictures can help you understand questions.
So if there's no picture in a question, you should draw one yourself.

Example 2

Imelda's brother is 56 inches tall. The top of his head comes to Imelda's shoulders. Write an expression for the height of Imelda.

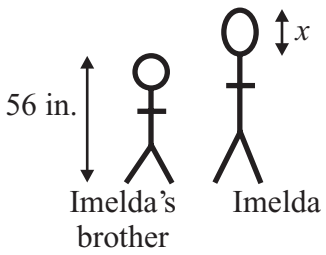
Solution

Draw a picture — a simple one will do fine.

The unknown length is the height of Imelda's head.

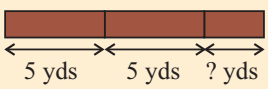
So choose a letter to represent it — x, say.

Then Imelda's height is **$(56 + x)$ inches**.



Guided Practice

1. Write an expression for the length of these boxes.



2. Susan's backyard has a swimming pool at the end. She measures the distance from the house to the swimming pool as 50 meters. By choosing a letter to represent the length of the swimming pool, write an expression for the length of the entire yard.

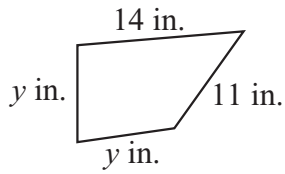
Perimeter Means the Distance Around a Shape

The **perimeter** is the distance **around the outside** of a shape.
You find a shape's perimeter by adding together the lengths of its sides.

If you don't know the length of a side, use a **variable**.

Example 3

Write an expression for the perimeter of the shape below.
The two unknown sides have the same length, y inches.



Solution

The perimeter is found by adding the lengths of all the sides together.

So, the perimeter of the shape is $11 + 14 + y + y$.
This can be simplified to **$(25 + 2y)$ inches**.

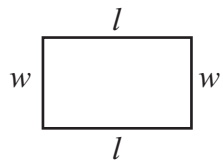
Don't forget:

You can simplify expressions by combining terms that contain the same variables.
So $7 + x + x + x$ can be simplified to $7 + 3x$.

When you write an expression for the perimeter of a **rectangle**, there could be **two** unknown lengths — the length and the width.
So you might need to use **two** variables.

Example 4

Write an expression for the perimeter of the rectangle on the right.



Solution

The perimeter of the shape is $w + l + w + l = \mathbf{2w + 2l}$.

Or you could write this expression as $\mathbf{2l + 2w}$.
It means exactly the same.

Check it out:

You use the associative and commutative properties all the time in math.
Most times, you probably do it almost without thinking — for example, when you work out $10 + 3$ instead of $3 + 10$ (an example of the commutative property of addition being used).
But now you need to be able to recognize when you do it. There'll be many examples pointed out throughout this book.

Add Numbers Together in Any Order

When you add the lengths together to find the perimeter of a shape, you can add numbers together **in any order**.

This is because addition is **commutative** and **associative**.

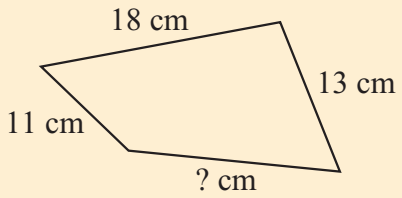
Commutative Property of Addition: $x + y = y + x$

Associative Property of Addition: $(x + y) + z = x + (y + z)$

Guided Practice

3. Find the perimeter of a rectangle with length 8 inches and width 5 inches.

4. Write and simplify an expression for the perimeter of the shape shown below.



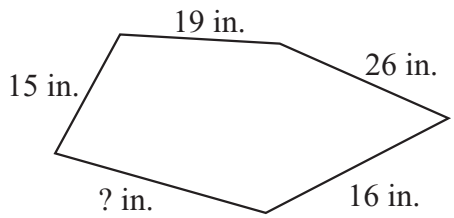
5. A triangle has one side of length 3 inches, one side of length 4 inches, and a third side of length q inches. Write and simplify an expression for the perimeter of this triangle.

Solve Geometry Equations by Manipulating Them

If you're given enough information, you can set up an equation to find a missing length or a perimeter.

Example 5

Write an expression for the perimeter of the shape below.
If the full perimeter is 100 inches, find the length of the unknown side.



Solution

The perimeter is the distance all the way around the outside.
Using the picture, this is $(15 + 19 + 26 + 16 + x)$ inches, where x is the length of the unknown side. This can be simplified to **$76 + x$ inches**.

Now you can write an equation using the other information in the question — the fact that the full perimeter is 100 inches.
So **$76 + x = 100$**.

You can then solve this equation by subtracting 76 from both sides.

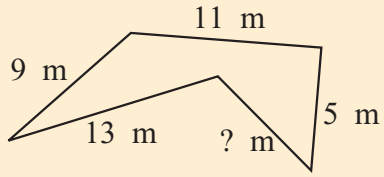
$x + 76 = 100$ **$76 + x$ is the same as $x + 76$**
$x = 100 - 76$ **Subtract 76 from both sides, and simplify.**
$x = 24$

So the length of the unknown side is **24 inches**.

Don't forget:
You could use the "guess and check" method to solve this equation.
But "guess and check" only works well for finding integers.

Guided Practice

6. By first writing an expression, find the unknown length on the shape shown below, which has a perimeter of 60 m.

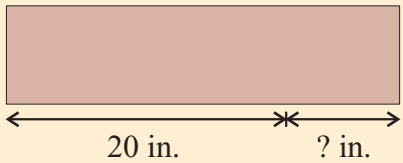


7. Liz knows that the perimeter of her kitchen is 50 yards. She knows from measuring that three of its four walls have a combined length of 32 yards. What is the length of the remaining wall?

Independent Practice

Now try these:
Lesson 2.3.1 additional questions — p434

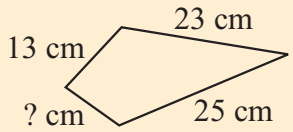
1. Write an expression for the length of the piece of wood shown below.



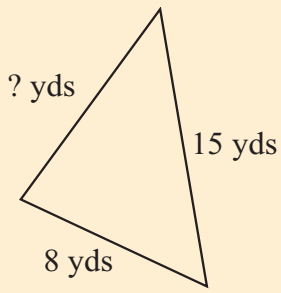
2. Use "guess and check" to find the unknown length in Exercise 1 if the total length of the wood is 32 inches.

Write an expression for the perimeter of each shape given in Exercises 3–4.

3.

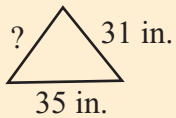


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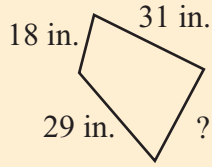


In Exercises 5–6, form and solve an equation to find the unknown length, given that the perimeter of each shape is 100 inches.

5.



6.



Round Up

Dealing with expressions about lengths isn't harder than dealing with expressions in any other situation. The main thing is to remember how to make an expression in the first place. Once you've mastered that, you should be ready to tackle area...

Lesson 2.3.2

Expressions About Area

California Standards:

Algebra and Functions 1.2

Write and evaluate an algebraic expression for a given situation, using up to three variables.

Algebra and Functions 1.3

Apply algebraic order of operations and the **commutative, associative**, and distributive **properties to evaluate expressions; and justify each step in the process.**

Algebra and Functions 3.1

Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ — the formulas for the perimeter of a rectangle, **the area of a triangle**, and the circumference of a circle, respectively).

Algebra and Functions 3.2

Express in symbolic form simple relationships arising from geometry.

What it means for you:

You'll practice using expressions to represent the area of objects.

Key words:

- length
- area
- expression
- commutative
- associative

Check it out:

The formula for the area of a rectangle:
"Area = base × height,"
is often shortened to just "$A = bh$."

You've used expressions to represent lengths and perimeters, but they can be used to represent just about anything else as well.

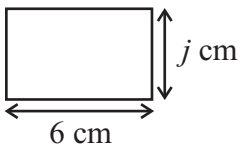
In this Lesson, they're used to describe areas. The good thing is that all the ideas about variables and solving equations work exactly the same as before.

Expressions Can Describe Area

You've already seen the formula for the area of a rectangle — **area = base × height**. But if you don't know one or both of these measurements, you have to use variables.

Example 1

Write an expression for the area of the rectangle on the right.
Then use it to find the area when $j = 4$.



Solution

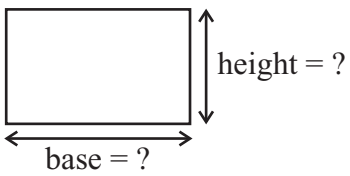
Area = base × height = $6 \times j = \mathbf{6j\ cm^2}$.

Evaluating this for $j = 4$ gives an area of $6 \times 4 = \mathbf{24\ cm^2}$.

If there are **two** unknowns, you need **two** variables.

Example 2

Write an expression for the area of the rectangle on the right.



Solution

Both the base and the height are unknown, so you need to use variables for both of them.

Call the height of the rectangle h, and the base b.

Then the area of the rectangle is $b \times h$, or $\boldsymbol{bh}$.

Guided Practice

1. Write an expression that represents the area of a rectangle with base b cm and height 4 cm.

2. Write an expression that represents the area of a rectangle with base 12 inches and height f inches.

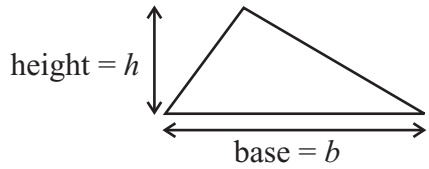
3. Evaluate your expression from Exercise 2 to find the area of the rectangle when the height is 3 inches.

The Area of a Triangle is Half the Area of a Rectangle

You've seen the formula for the area of a triangle before: **area** $= \frac{1}{2} \times$ **base** $\times$ **height**. Just like with rectangles, if you **don't know** the length of the base or the height, you'll need to use a **variable**.

Example 3

Write an equation describing the area of the triangle shown below. Then use your equation to find the triangle's area if its height is 3 inches and its base is 8 inches.



Solution

There are two unknowns, so your expression will need to include two variables — the height h and the base b.

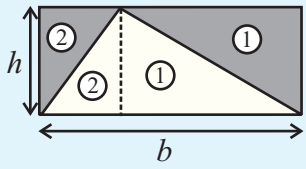
Use these variables in the triangle area formula: $\text{Area} = \frac{1}{2} \times b \times h = \frac{1}{2}bh$

Use this equation to find the triangle's area when $b = 8$ and $h = 3$.

$\text{Area} = \frac{1}{2}bh = \frac{1}{2} \times 8 \times 3 = 12 \text{ in}^2$

Check it out:

Notice that the area of a triangle is half the area of the rectangle with the same base and height measurements.



In this diagram, the areas marked 1 are equal, and the areas marked 2 are equal. The white triangle has half the area of the rectangle.

Check it out:

The formula for the area of a triangle is usually shortened to just "$A = \frac{1}{2}bh$."

Multiply Numbers Together in Any Order

Like addition, multiplication is also **commutative** and **associative**.

Commutative Property of Multiplication: $xy = yx$

So the area of the rectangle in Example 2 could have been written hb.

Associative Property of Multiplication: $(xy)z = x(yz)$

This means you don't need parentheses in the formula for the area of a triangle to show the order of the multiplications.

$\left(\frac{1}{2} \times b\right) \times h$ is the same as $\frac{1}{2} \times (b \times h)$.

Check it out:

As with the equivalent properties for addition in the previous lesson, you probably use the associative and commutative properties for multiplication all the time without thinking — for example, when you work out 2 × 9 instead of 9 × 2 (this uses the commutative property of multiplication). Examples of these properties being used will be shown throughout this book.

Guided Practice

4. Write an expression that represents the area of a triangle of height 2 inches and base y inches.

5. Evaluate your expression from Exercise 4 to find the area of the triangle when the base is 7 inches.

6. Write an expression that represents the area of a triangle with base x cm and height y cm.

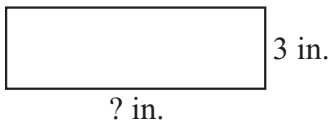
Use Equations to Find Missing Lengths

Equations can be used to find missing lengths.

First you have to write an **equation** to **describe** the situation.
Then you can **solve** it to find the missing value.

Example 4

The area of the rectangle shown is 36 in^2.
Find the length of the unknown side.



Solution

First write an equation linking the different quantities in the question.
These are:
- the side you know the length of (3 in.),
- the side whose length you need to find out — call this b,
- the area of the triangle (36 in^2).

The equation linking these three quantities is:
area = base × height, or $A = bh$.
Substituting in the values you know, this gives $36 = 3 \times b$, or $\mathbf{3b = 36}$.

Now solve this equation: $3b = 36$

$b = 12$ **Divide both sides by 12.**

This means the length of the unknown side is **12 inches**.

Don't forget:
The method for solving equations like these was explained in Lesson 2.2.4.

Guided Practice

7. A rectangle has area 12 in^2 and its base is 3 inches.
Form and solve an equation to find its height.

8. A triangle has area 25 in^2 and its height is 5 inches.
Form and solve an equation to find the length of its base.

Independent Practice

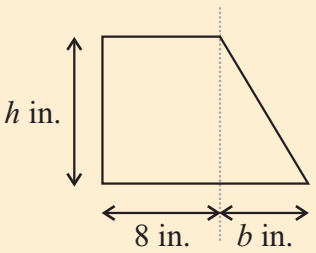
Now try these:
Lesson 2.3.2 additional questions — p435

1. Write an expression for the area of a square with side length b cm.

2. Write an expression for the area of a triangle with base 6 inches and height h inches.

3. Evaluate your expression from Exercise 2 to find the area of the triangle when the height is 3 inches.

4. By breaking it into simple shapes, write an expression for the area of the shape shown to the right.



Round Up

Expressions for areas of triangles have that tricky ½ — so they can be awkward. Focus on working out each little bit of the expression in turn — that's the best way to avoid mistakes.

Lesson 2.3.3

Finding Complex Areas

California Standards:

Algebra and Functions 3.1
Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, **$A = \frac{1}{2}bh$**, $C = \pi d$ — the formulas for the perimeter of a rectangle, **the area of a triangle**, and the circumference of a circle, respectively).

Mathematical Reasoning 1.3
Determine when and how to break a problem into simpler parts.

What it means for you:
You'll see how you can use the formulas for the areas of rectangles and triangles to find areas of much more complex shapes too.

Key words:
- complex shape

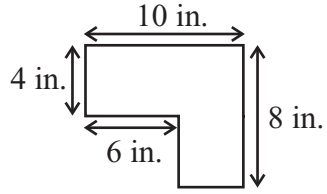
Check it out:
There's often more than one way you can break a complicated shape up. Try to spot the way that will make your calculations as easy as possible.

Finding the area of a rectangle or a triangle is one thing. But once you can do that, you can start to find out the areas of some really complicated shapes using those very same techniques.

This is an important idea in math — using what you know about simple situations to find out about more complex ones.

Find Complex Areas by Breaking the Shape Up

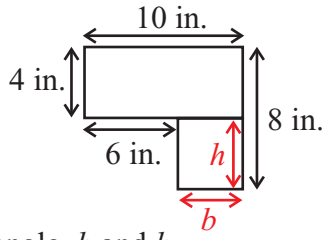
There's no easy formula for working out the area of a shape like this one.



But you can find the area by **breaking the shape up** into two smaller rectangles.

Example 1

Find the area of the shape above.



Solution

Divide the shape into two rectangles, as shown.

Area of large rectangle = $10 \times 4 = \mathbf{40\ in^2}$.

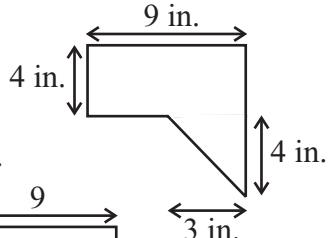
Now you need the dimensions of the small rectangle, b and h.
$b = 10 - 6 = \mathbf{4\ in.}$ And $h = 8 - 4 = \mathbf{4\ in}$.
So the area of the small rectangle = $bh = 4 \times 4 = \mathbf{16\ in^2}$.

So the total area of the shape is $40 + 16 = \mathbf{56\ in^2}$.

You don't always have to break a complicated shape down into rectangles. You just have to break it down into simple shapes that you know how to find the area of.

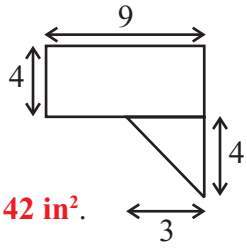
Example 2

Find the area of the shape on the right.



Solution

Divide the shape into a rectangle and a triangle.



Area of rectangle = $9 \times 4 = 36\ \text{in}^2$.

Area of triangle = $\frac{1}{2} \times 4 \times 3 = 6\ \text{in}^2$.

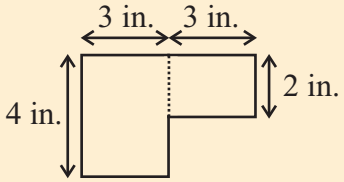
So the total area of the shape is $36 + 6 = \mathbf{42\ in^2}$.

Guided Practice

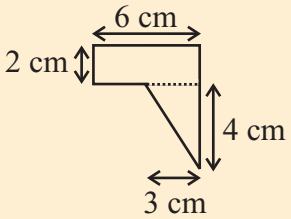
Check it out:
These problems can look tough, so take it step by step. Work out the area of each rectangle or triangle in the shape in turn.
Don't try to do the whole calculation at once.

Find the areas of the shapes below.

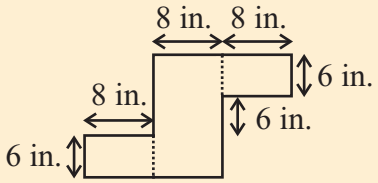
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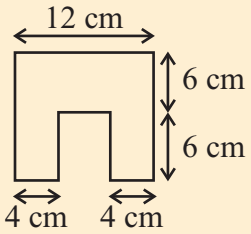
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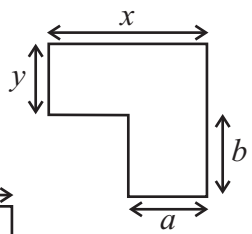


Complex Areas Can Involve Variables

Sometimes you have to use variables for the unknown lengths.
But you can write an expression in just the same way.

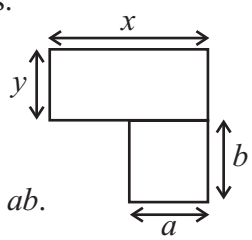
Example 3

Find the area of the shape on the right.



Solution

Divide the shape into two rectangles.



Area of the large rectangle = xy.
Area of the small rectangle = ab.

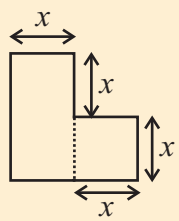
So the total area of the shape is $xy + ab$.

Guided Practice

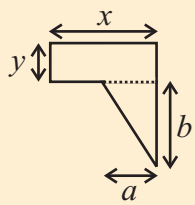
Check it out:
If you're finding questions 6–8 tough, then look back at Exercises 2–4 above. The shapes are exactly the same, only this time the lengths are variables instead of numbers. Work out the areas in just the same way.

Find the areas of the shapes below.

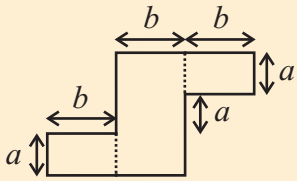
5.



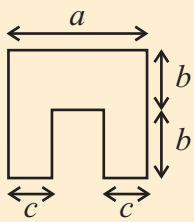
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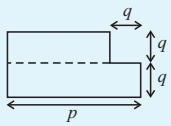


8.



Check it out:

You can divide the shape up differently and get the same answer. For example, you could divide it into two rectangles:



Then the total area is $pq + q(p - q)$.
Using the distributive property (covered in the next lesson), you'll see that this gives exactly the same answer as in Example 4.

$pq + q(p - q) = pq + qp - q^2$
$= 2pq - q^2$

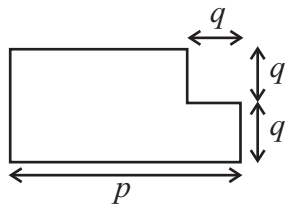
Look for another way to divide up the shape, and check that it gives the same expression for the area.

You Can Subtract Areas As Well

Sometimes it's easier to find the area of a shape that's too big, and subtract a smaller area from it.

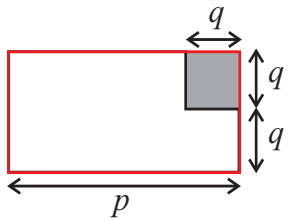
Example 4

Calculate the area of the shape on the right.



Solution

This time it's easier to work out the area of the rectangle with the red outline, and subtract the area of the gray square.



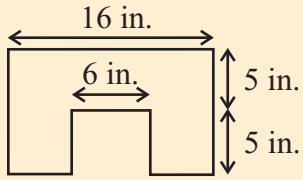
Area of red rectangle $= p \times 2q = 2pq$.
Area of gray square $= q \times q = q^2$.

So area of original shape $= \mathbf{2pq - q^2}$.

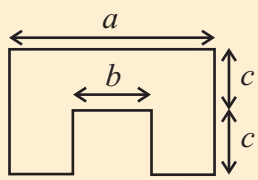
Guided Practice

Use subtraction to find the areas of the shapes below.

9.



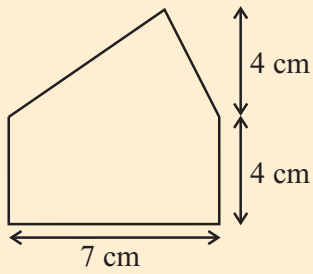
10.



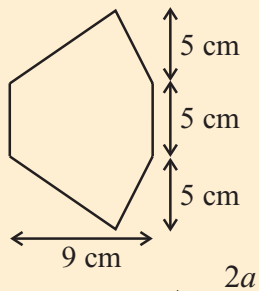
Independent Practice

Find the areas of the shapes below.

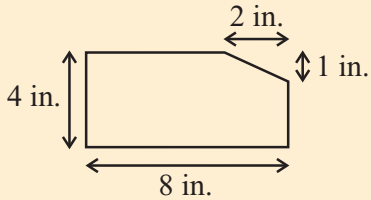
1.



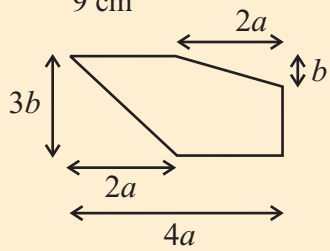
2.



3.



4.



Now try these:

Lesson 2.3.3 additional questions — p435

Round Up

Remember that it doesn't matter whether your lengths are numbers or variables — you treat the problems in exactly the same way. That's one of the most important things to learn in algebra.

Lesson 2.3.4

The Distributive Property

California Standard:

Algebra and Functions 1.3

Apply algebraic order of operations and the commutative, associative, and **distributive properties to evaluate expressions; and justify each step in the process.**

What it means for you:

You'll learn about the distributive property, which allows certain types of multiplication to be rewritten in other ways and helps simplify mental math.

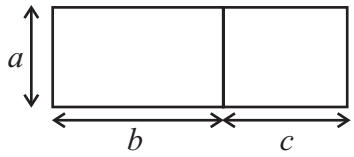
Key words:

- distributive property
- multiplication
- parentheses

The distributive property is the name given to a way of breaking up certain types of multiplication. It can help a lot with math once you understand it properly.

Use the Distributive Property to Remove Parentheses

Look at the shape on the right.



You can work out its total area in two different ways.

- You can find the areas of the two smaller rectangles and add them. Total area = area of 1st rectangle + area of 2nd rectangle = $ab + ac$.
- Or you can find the total width of the whole rectangle by adding b and c, and then multiply by the height. Total area = $a(b + c)$.

But these two expressions must be the same, since they both represent the same area. This is an example of the **distributive property**.

Distributive Property: $a(b + c) = ab + ac$

Check it out:

"Verify" means "Check that something is true."
So here, you just put the numbers in the distributive property equation and check that it "works."

Example 1

Verify the distributive property in the case where $a = 7$, $b = 5$, and $c = 3$.

Solution

- First work out $ab + ac$. This is $(7 \times 5) + (7 \times 3) = 35 + 21 = \mathbf{56}$
- Next work out $a(b + c)$. This is $7 \times (5 + 3) = 7 \times 8 = \mathbf{56}$

Both sides are equal, and so the distributive property holds.

The distributive property means you can **remove parentheses** from an expression.

Example 2

Remove the parentheses from the expression $p(q + r)$.

Solution

Using the distributive property:

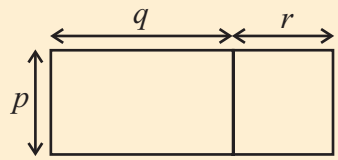
$$p(q + r) = pq + pr$$

Check it out:

For an alternative way to do this Example, see Exercise 1 in the next Guided Practice.

Guided Practice

1. Verify the distributive property by calculating the area of this rectangle in two different ways.



Draw a picture as in Exercise 1 to represent the expressions in Exercises 2–5. In each case, verify the distributive property by finding the area in two ways.

2. $2 \times (7 + 9)$ **3.** $20 \times (9 + 4)$

4. $h \times (3 + h)$, for $h = 10$ **5.** $g \times (y + d)$, for $g = 3$, $y = 9$, $d = 5$.

Remove the parentheses from the expressions in Exercises 6–8.

6. $p \times (q + r)$ **7.** $r \times (s + t)$ **8.** $b \times (c + d)$

Remove the parentheses from the expressions in Exercises 9–11.

9. $(q + r) \times m$ **10.** $(a + b) \times c$ **11.** $(e + f) \times g$

Don't forget:
$a \times (b + c)$
is exactly the same as
$a(b + c)$.

Don't forget:
Using the commutative property of addition, you can write $(q + r) \times m$ as $m \times (q + r)$.

The Distributive Property Works with Subtraction Too

If the parentheses contain a **subtraction**, then when you remove the parentheses, the expression will still contain a subtraction.

$$a(b - c) = ab - ac$$

Example 3

Verify the distributive property for $2 \times (6.5 - 0.5)$.

Solution

$(2 \times 6.5) - (2 \times 0.5) = 13 - 1 = \mathbf{12}$

$2 \times (6.5 - 0.5) = 2 \times 6 = \mathbf{12}$

This rule works for **any** numbers — even **negative** ones.
Remember to be extra careful if **negative numbers** are involved.

Example 4

Verify the distributive property for $-2 \times (6.5 - 0.5)$.

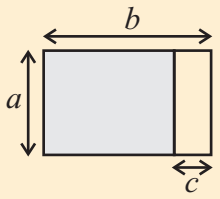
Solution

$(-2 \times 6.5) - (-2 \times 0.5) = -13 - (-1) = -13 + 1 = \mathbf{-12}$

$-2 \times (6.5 - 0.5) = -2 \times 6 = \mathbf{-12}$

Guided Practice

12. Verify the distributive property equation $a(b - c) = ab - ac$ by calculating the area of the shaded rectangle in two different ways.



Use the distributive property to evaluate the expressions in Exercises 13–20. Show your work.

13. $4 \times (8 - 2)$
14. $9 \times (15 - 8)$
15. $-5 \times (-7 - 5)$
16. $9 \times (8 - 15)$
17. $-6(23 - 18)$
18. $-2(-2 - 6)$
19. $3 \times (17 - p)$, given that $p = 7$
20. $m \times (n - p)$, given that $m = 8$, $n = 7$, $p = 5$

Remove the parentheses from the expressions in Exercises 21–23.

21. $r \times (s - t)$
22. $a \times (b - c)$
23. $m \times (n - p)$

Say whether each of Exercises 24–27 is true or false.

24. $y \times (2 - 3) = y \times 2 - y \times 3$
25. $f \times (g - u) + y = f \times g + f \times u + f \times y$
26. $k \times (1 - 12) = k - 12$
27. $t \times 9 = t \times 10 - t$

Independent Practice

Use the distributive property to evaluate the expressions in Exercises 1–8. Show your work.

1. $2 \times (6 - 3)$
2. $4 \times (22 - 14)$
3. $-4 \times (-2 - 8)$
4. $2 \times (6 - 23)$
5. $-6 \times (-23 + 8)$
6. $-3 \times (-8 - 1)$
7. $q \times (17 - q)$, given that $q = 3$
8. $b \times (c - d)$, given that $b = 10$, $c = 6$, $d = 8$

Remove the parentheses from the expressions in Exercises 9–11.

9. $t \times (d - r)$
10. $(y - h) \times z$
11. $(9 - s) \times 3$

Say whether each of Exercises 12–15 is true or false.

12. $v \times (5 - e) = 5v - 5e$
13. $p \times (q - 3) + 7 = pq - 3p + 7p$
14. $(2 - m) \times n = 2n - mn$
15. $-d \times (-9 - r) = 9d + dr$

Now try these:
Lesson 2.3.4 additional questions — p436

Round Up

The distributive property is very useful in math — it comes up again and again. So learn it now... Practice writing it down a few times until you can do it without thinking. It'll definitely be useful in the next Lesson.

Lesson 2.3.5

Using the Distributive Property

California Standard:

Algebra and Functions 1.3

Apply algebraic order of operations and the commutative, associative, and **distributive properties to evaluate expressions; and justify each step in the process.**

What it means for you:

You'll learn how to use the distributive property for mental math problems and problems involving patterns.

Key words:

- distributive property
- multiplication
- expression

The distributive property is incredibly useful — not only when you're doing algebra, but also when you're trying to work out complicated sums in your head.

The Distributive Property Makes Mental Math Easier

The distributive property can help with mental math.

The idea is to split complicated numbers into numbers that are easy to multiply in your head.

Example 1

Calculate 7×14 without using a calculator.

Solution

$$\begin{aligned} 7 \times 14 &= 7 \times (10 + 4) && \text{Write 14 as 10 + 4} \\ &= (7 \times 10) + (7 \times 4) && \text{Use the distributive property} \\ &= 70 + 28 = \mathbf{98} \end{aligned}$$

Example 2

Calculate 5×107 without using a calculator.

Solution

$$\begin{aligned} 5 \times 107 &= 5 \times (100 + 7) && \text{Write 107 as 100 + 7} \\ &= (5 \times 100) + (5 \times 7) && \text{Use the distributive property} \\ &= 500 + 35 = \mathbf{535} \end{aligned}$$

Sometimes it's easier to use a subtraction.

Example 3

Calculate 6×29 without using a calculator.

Solution

$$\begin{aligned} 6 \times 29 &= 6 \times (30 - 1) && \text{Write 29 as 30 – 1} \\ &= (6 \times 30) - (6 \times 1) && \text{Use the distributive property} \\ &= 180 - 6 = \mathbf{174} \end{aligned}$$

Guided Practice

Using the distributive property, do Exercises 1–8 as mental math.

1. 6×13
2. 7×21
3. 2×17
4. 21×8
5. 11×13
6. 16×5
7. 3×33
8. 15×9

Don't forget:

Use the commutative property of multiplication to rewrite these Exercises in the easiest order. For example, 21×8 is the same as 8×21.

The Distributive Property Is Used All the Time

The distributive property can be used to rewrite expressions from all areas of math.

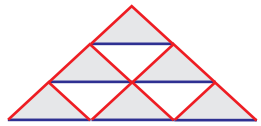
Example 4

This triangle is made out of 2 red sticks and 1 blue stick.



The total length of the sticks in one triangle is given by $2r + b$, where r is the length of a red stick and b is the length of a blue stick.

Write an expression for the total length of sticks in the pattern on the right.



Solution

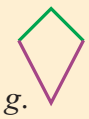
The pattern is made up of 6 of the small triangles.
So the total length of red and blue sticks is $\mathbf{6 \times (2r + b)}$.

Using the distributive property, this can be rewritten:
$6 \times (2r + b) = (6 \times 2r) + (6 \times b) = \mathbf{12r + 6b}$

Don't forget:
It's always a good idea to check your answers. Here, you could count the number of red and blue sticks in the final diagram, and see if it matches your expression.

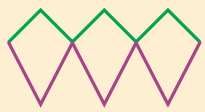
Guided Practice

Exercises 9–11 are about the diagram on the right.



9. Write an expression for the total length of the sides, if each purple side has length p and each green side has length g.

10. Write an expression for the total length of the sides, if 3 of these shapes are arranged in a pattern as shown. Write your expression in two different ways, using the distributive property.



11. Write two expressions for the length of the sides if 100 of the shapes are arranged in a similar pattern.

Independent Practice

1. Carl is at the store to buy school supplies. Pens cost \$1.15 each and a package of computer paper costs \$3.75. He wants to buy two pens and two packages of paper. Write an expression of the form $a \times (b + c)$ that can be used to represent this situation.

2. Use the distributive property to calculate the money Carl spent.

Use the distributive property to do Exercises 3–8 as mental math.

3. 9×29
4. 14×12
5. 19×7
6. 17×7
7. 15×11
8. 9×31

Rewrite Exercises 9–12 using the distributive property.

9. $a \times (11 - b)$
10. $j \times (g + k)$
11. $7 \times (3 + s)$
12. $15 \times (q - w)$

Now try these:
Lesson 2.3.5 additional questions — p436

Round Up

This mental math technique might seem hard to begin with, but can be really useful once you've had some practice. And if you can manage it, it's a trick that'll impress all your friends.

Lesson 2.3.6

Squares and Cubes

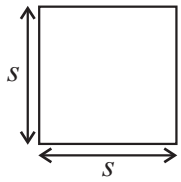
This is mostly a Lesson on things you've seen before — expressions, equations, order of operations... But everything comes about from looking at (and thinking about) squares and cubes.

Squares Are a Special Sort of Rectangle

You've already seen that you can write the formula for the area of a rectangle as $A = bh$.

But a square is just a rectangle whose sides all have the same length — call it s.
So the formula for the area becomes:

$$A = s \times s \quad \text{or} \quad A = s^2$$



Remember — s^2 ("s squared") means exactly the same as $s \times s$.

Example 1

Write an expression for the total area of 4 identical squares with sides of length d.

Evaluate your expression in the case where $d = 3$ in.

Solution

The area of one square is given by d^2.
So the area of 4 identical squares is given by $\mathbf{4d^2}$.

You now have to evaluate this expression for $d = 3$.
The total area will be $\mathbf{4 \times 3^2}$ — but remember to evaluate exponents before multiplication.
So the total area is $4 \times 3^2 = 4 \times 9 = \mathbf{36\ in^2}$.

You can use "guess and check" to find the length of a side if you know the area.

Example 2

A square has area 36 in^2. What is the length of its sides?

Solution

You need to solve the equation $36 = s^2$. That means you need to find a value for s that gives 36 when you square it.

Use "guess and check":
- Try $s = 10$. Here s^2 comes to $10 \times 10 = 100$, which is much too large.
- Try $s = 5$. Now s^2 comes to $5 \times 5 = 25$, which is too small.
- Try $s = 6$. Now s^2 comes to $6 \times 6 = 36$.

So the square must have sides of length **6 inches**.

California Standards:

Algebra and Functions 1.2
Write and evaluate an algebraic expression for a given situation, using up to three variables.

Algebra and Functions 3.1
Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ — the formulas for the perimeter of a rectangle, the area of a triangle, and the circumference of a circle, respectively).

Algebra and Functions 3.2
Express in symbolic form simple relationships arising from geometry.

What it means for you:

You'll practice using expressions to represent areas and volumes of objects.

Key words:

- square
- cube
- edge
- face
- expression
- power
- area
- volume

Don't forget:

Evaluate exponents before doing multiplication — see PEMDAS in Lesson 2.1.4.

Don't forget:

3^2 can be said out loud as either "3 squared" or "3 to the second power." Both mean exactly the same thing.

Guided Practice

1. Write an expression for the area of a square with side length z.

2. A square has area 81 cm^2. What is the length of its sides?

3. A square has area 49 in^2. What is the length of its sides?

4. A square has area 16 yd^2. What is the length of its sides?

5. A square has area 64 m^2. What is the length of its sides?

6. A square has area 121 in^2. What is the length of its sides?

Don't forget:
Units such as in^2 should be read as "square inches."

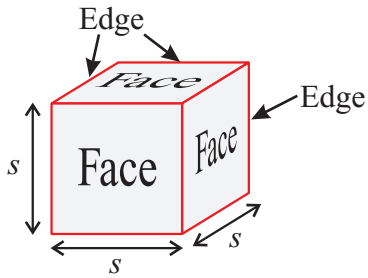
Check it out:
The straight lines that run along the side of faces are known as edges.
They are shown red on the diagram on the right — 9 are visible, but there are 12 altogether.
See Section 7.4 for more information.

Cubes Are "Three-Dimensional Squares"

A cube is a three-dimensional solid with six square faces.

Its **surface area** is the total area of all 6 of its faces. As each face is a **square**, the formula for surface area of a cube is:

$$A = 6s^2$$



Example 3

Without using a calculator, find the surface area of a cube whose edges have length 9 cm.

Solution

The surface area of the cube is given by $6s^2$.

If $s = 9$, this is $6 \times 9^2 = 6 \times 81$ — **Calculate the exponent first.**

$= 6 \times (80 + 1)$ — **Rewrite 81 as 80 + 1.**

$= 6 \times 80 + 6 \times 1$ — **Use the distributive property.**

$= 480 + 6$ — **Evaluate × before +.**

$= \mathbf{486\ cm^2}$.

Or you might have to **solve equations** to do with cubes.

Example 4

The surface area of a cube is 54 in^2.
Use "guess and check" to find the length of the cube's edges.

Solution

The surface area of the cube is given by the expression $6s^2$.
You need to find s such that $6s^2 = 54$. Use "guess and check."

- Try $s = 4$. Here $6s^2$ comes to $6 \times 4^2 = 6 \times 16 = 96$, which is too large.
- Try $s = 2$. Now $6s^2$ comes to $6 \times 2^2 = 6 \times 4 = 24$, which is too small.
- Try $s = 3$. Now $6s^2$ comes to $6 \times 3^2 = 6 \times 9 = 54$, which is perfect.

So the lengths of the cube's edges must be **3 inches**.

Guided Practice

7. Write an expression for the surface area of a cube with edge length k.

8. A cube has surface area 96 in^2. What is the length of its edges?

9. A cube has surface area 216 cm^2. What is the length of its edges?

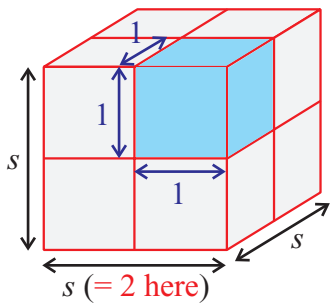
10. A cube has surface area 150 yd^2.
What is the length of its edges?

Volume Is a Measurement of 3-D Space

Volume is a measurement of the **amount of space inside** a **three-dimensional** object.

It's measured in **cubic units** and equals the number of **unit cubes** (cubes whose edges have length 1) that fit inside the object.

In the diagram on the right, each side has a length of **2 units**, so two unit cubes fit along each side. (One unit cube is shaded **blue**.)



You can calculate the volume of a cube using the formula:

$$V = s \times s \times s \quad \text{or} \quad V = s^3$$

Check it out:

The volume of a unit cube is always 1 cubic unit (1 unit^3). This is because, for a unit cube, $s = 1$, and so $s^3 = 1 \times 1 \times 1 = 1$.

But notice that if you use different units, your unit cube changes too. For example, if you're using yards, then your unit cube has sides of length 1 yard, and so its volume is 1 yd^3. However, if you're measuring things in centimeters, then a unit cube will have sides of length 1 cm, and a volume of 1 cm^3.

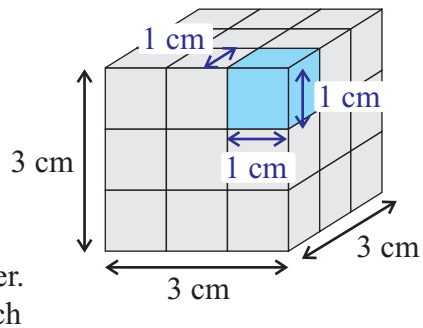
Example 5

Draw a picture to show the unit cubes in a cube with edge length 3 cm. Find the volume of the cube by counting unit cubes.

Then verify your answer using the equation $V = s^3$.

Solution

The edges have length 3 cm, so 3 unit cubes (using units of cm) will fit along each edge.



Count the unit cubes in layers, starting at the front.
There are 9 unit cubes in the front layer.
And there are 3 layers altogether, which means there are $9 \times 3 = \mathbf{27}$ **unit cubes** altogether. And since each unit cube has a volume of 1 cm^3, the volume must be **27 cm^3**.

Now use the equation $V = s^3$ with $s = 3$ to check the result.
$V = 3^3 = \mathbf{3 \times 3 \times 3 = 27}$ **cm^3**.

Guided Practice

11. Using the formula, find the volume of a cube with edges 6 meters long.

12. By counting the number of unit cubes inside, work out the volume of a cube with edges 3 inches long.

13. Use "guess and check" to find the edge length of a cube with a volume of 64 cubic yards.

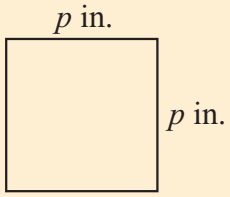
Independent Practice

1. Find the area of a square whose sides have length 5 in.

2. Find the volume of a cube whose sides have length 7 in.

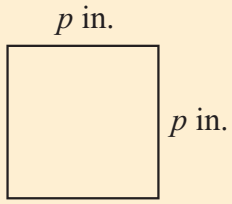
In Exercises 3–7, find the value of p.

3.



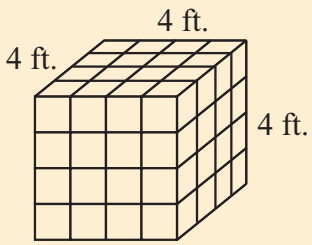
Area = 36 in^2

4.



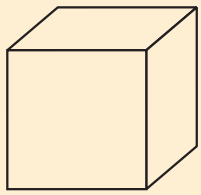
Area = 144 in^2

5.



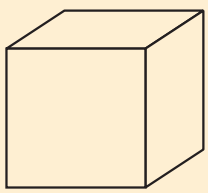
Volume = p ft^3

6.



Edge Length = p cm
Surface Area = 150 cm^2

7.



Edge Length = p in.
Volume = 729 in^3

Check it out:
Another way to write 24 in^2 is 24 sq. in.
They both mean "24 square inches."

Now try these:
Lesson 2.3.6 additional questions — p436

Round Up

There were a few formulas in this Lesson, but in the end it always comes down to two skills — being able to substitute values into a formula, and being able to work backward to find out what values need to be substituted. Keep practicing those skills.

Lesson 2.3.7

Expressions and Angles

California Standards:

Algebra and Functions 1.2

Write and evaluate an algebraic expression for a given situation, using up to three variables.

Algebra and Functions 3.1

Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ — the formulas for the perimeter of a rectangle, the area of a triangle, and the circumference of a circle, respectively).

Algebra and Functions 3.2

Express in symbolic form simple relationships arising from geometry.

What it means for you:
More expressions practice — this time with angles.

Key words:
- angle
- triangle
- expression

Don't forget:
The little square symbol in the corner of the angle MED (□) means that corner is a right angle. A right angle has a measure of 90°.

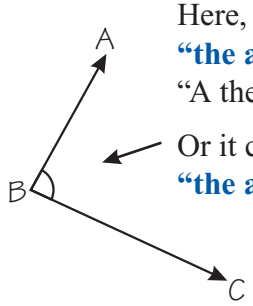
More expressions, but this time about angles. Some of this you'll have seen last year, but now you can apply your knowledge of expressions.

As with the earlier Lessons, expressions are used as a shorthand way of explaining how to work something out. In the world of angles, that usually means many pictures of triangles...

Angles Are Measured Between Two Lines

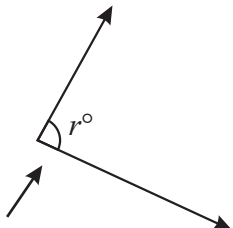
Whenever two lines meet, you get an angle.
The first thing you need to know about angles is how to label them.

The diagrams below show two different ways of identifying angles:



Here, the marked angle could be called **"the angle ABC."** If you follow the path "A then B then C," you trace out the angle.

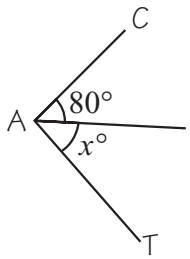
Or it could simply be called **"the angle at B."**



On this diagram, the size (or measure) of the angle has been labeled with the variable ***r***.

Example 1

Write an expression for the measure of angle CAT.



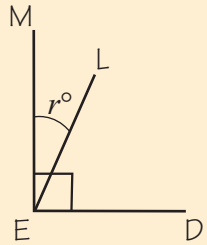
Solution

The angle CAT is made up of two parts — an 80° angle and then a further angle of $x°$.

So, the angle CAT must be $(80 + x)°$.

Guided Practice

Exercises 1–2 make use of the diagram on the right.



1. Write an expression for the measure of angle LED if MED = 90° and MEL = $r°$.

2. Evaluate your expression to find the measure of angle LED when $r = 30$.

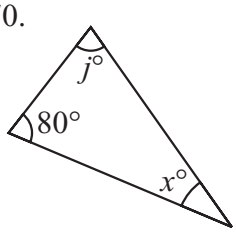
The Angles in a Triangle Always Add Up to 180°

The measures of the angles in a triangle always add up to 180°.
You can use this fact to make expressions for unknown angles.

Check it out:
There's a lot more about angles in triangles in Section 7.2. But for now, just remember that the measures of the angles in a triangle add up to 180°.

Example 2

Use the diagram below to write an equation involving x.
Solve it, given that $j = 70$.



Check it out:
There are other correct equations you could have, but they would all be equivalent to $j + x = 100$.
For example, you could have:
$80 + j + x = 180$
$x = 100 - j$

Solution

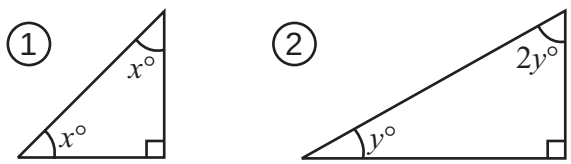
Use the fact that the angle measures in a triangle add up to 180°.

In the form of an equation, this is: $80 + j + x = 180$
Or, subtracting 80 from both sides, you get: $\boldsymbol{j + x = 100}$

To solve this, put $j = 70$.
Then your equation becomes: $70 + x = 100$
Now you can subtract 70 from both sides to get $\boldsymbol{x = 30°}$.

Example 3

Find the values of x and y in the diagrams below.



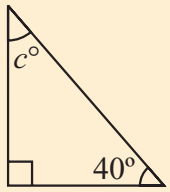
Solution

There's a right angle (90°) in the first triangle, so $x + x + 90 = 180$.
This means: $x + x = 90$, or $2x = 90$.
Divide both sides of this equation by 2 to get $x = 45$.
This means the two equal angles in the first triangle are **45°**.

There's also a right angle (90°) in the second triangle,
so $y + 2y + 90 = 180$.
This tells you: $2y + y = 90$, or $3y = 90$
Divide both sides of this equation by 3 to get $y = 30$.
So the unknown angles in the second triangle measure **30°** and **60°**.

Guided Practice

3. Write an equation for c in the triangle on the right. Then use your equation to find c.



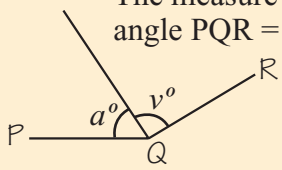
4. Mark draws a triangle and starts measuring the angles. The first angle he measures is 70°. He labels the other two angles' measures q and w. Write an equation involving q and w.

5. Mark measures w and finds out that it is 50. Use your expression from Exercise 4 to find the value of q.

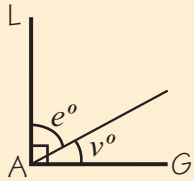
Independent Practice

Write an equation for v in each of Exercises 1–2.

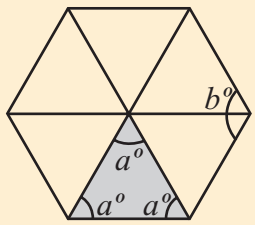
1. The measure of angle PQR = 120°



2.



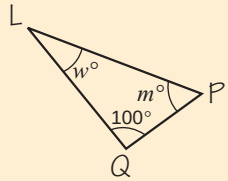
Exercises 3–4 are about this regular hexagon.



3. By considering the shaded triangle, write an equation containing a. Use your equation to find a.

4. Write an equation linking a and b. Use this equation and your answer to Exercise 3 to find the value of b.

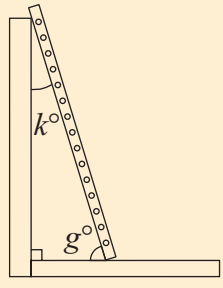
Exercises 5–6 are about the triangle on the right.



5. Omar measures the angle LQP and finds that it is 100°. He next measures the angle QLP, and calls it w. Write an equation for m, the measure of angle LPQ.

6. Omar finds that $w = 20°$. Use your answer to Exercise 5 to find the size of angle LPQ.

Exercises 7–8 are about the ladder shown below.



7. Jasmine leans a ladder against a wall, and sees a right triangle. She measures the angle the ladder makes with the ground, $g°$. Write an equation for k in terms of g.

8. The angle with measure $g°$ turned out to be 75°. Use your equation from Exercise 7 to find k.

Now try these:
Lesson 2.3.7 additional questions — p437

Round Up

The facts about angles inside a triangle often provide an excuse for some expression work. The technique is the same as for every other expression question — make an expression then you'll probably have to work something out using it.

Lesson 2.4.1

Section 2.4 Analyzing Problems

Part of math is being able to apply it to the real world. That means spotting what math to use and when to use it. Sometimes problems can be reduced to equations, other times you just need to think logically.

California Standards:

Mathematical Reasoning 1.1

Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, identifying missing information, **sequencing** and prioritizing **information, and observing patterns**.

Mathematical Reasoning 2.4

Use a variety of methods, such as words, **numbers, symbols,** charts, graphs, tables, diagrams, and models, **to explain mathematical reasoning.**

What it means for you:

You'll practice two of the fundamental math skills — being able to identify what is happening in different situations and how to use the information you're given.

Key words:

- pattern
- sequence
- relationship

Check it out:

This is an example of two patterns being combined. The first pattern is that the square rotates, so that the dots are on the next side in the clockwise direction. The second pattern is that the number of dots alternates — 1, then 2, then 1, and so on.

Some Problems Involve Patterns

When analyzing a problem, you need to try to figure out what is happening. Sometimes this means spotting a **pattern** in the results.

Example 1

Find the next pattern in this sequence:



Solution

The shape is **rotated clockwise** by 90° every time.

So the next one is .

Example 2

Find the next pattern in this sequence:



Solution

The pattern **alternates** between squares with two dots and squares with one dot. So the next square should have two dots.

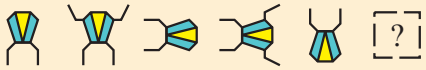
Every two squares, there is a **rotation clockwise by 90°**, so the next square should have its dots along the left edge of the square.

So the next one is .

Guided Practice

In Exercises 1–2, find the next shape in the pattern.

1. 

2. 

Patterns Can Be Number Based

Patterns don't just crop up in questions with pictures — they can occur in questions with **numbers** too.

Example 3

Find the next number in this sequence:

1, 3, 5, 7, 9, 11, ...

Solution

Each number is 2 more than the one before it.
So the next number must be **13**.

Example 4

Find the next number in this sequence:

1, 3, 6, 10, 15, ...

Solution

Between the first number and the second, 2 is added.

Between the second number and the third, 3 is added.

Between the third number and the fourth, 4 is added.

Between the fourth number and the fifth, 5 is added.

So, you should add 6 to get from the fifth number to the sixth.
That means that the sixth number is 15 + 6 = **21**.

Check it out:
It's always worth looking at the differences between consecutive numbers in a sequence like this.
Write the differences down — it can make a pattern easier to spot.
So in Example 4, you'd write down 2, 3, 4, 5...

Guided Practice

In Exercises 3–10, find the next number in the pattern.

3. 1, 2, 3, 4, 5, 6, ...
4. 10, 9, 8, 7, 6, ...
5. 4, 6, 8, 10, 12, ...
6. 1, 6, 11, 16, 21, ...
7. 1, 11, 111, 1111, 11,111, ...
8. 40, 35, 30, 25, 20, 15, ...
9. 1, 16, 161, 1616, 16,161, ...
10. 1, 2, 4, 7, 11, 16, ...

Check it out:
Sometimes it's not the value of the numbers that's important, but the pattern they make. Look at Exercise 9, for example.

Some Problems Ask Things in a Strange Order

Some problems might **look complicated**, even though the math behind them is actually quite simple.

Example 5

On his birthday, Scott wonders how old the public library was the year he was born. Scott is now 12 and he knows that the public library was 50 years old, 9 years ago. How old was the library when Scott was born?

Solution

The question gives you two pieces of information about two different times:

- **Now**: Scott is 12.
- **9 years ago**: The library was 50 years old.

You need to work out how old the library was when Scott was **born**. From your first piece of information, you can see this was **12 years ago**.

You now need to work out how old the library was 12 years ago. From the second piece of information, you can see it was 50 years old 9 years ago. So 12 years ago it must have been **47 years old**.

Check it out:

The only math this question asks you to do is adding and subtracting integers. But the way it asks the question is very complicated — and you need to think about it very logically and carefully.

Guided Practice

11. Eduardo needs another $17 to be able to afford a new video game that costs $40. Yesterday he spent $10 on comic books. How much did he have before he bought the comic books?

Independent Practice

Now try these:

Lesson 2.4.1 additional questions — p437

In Exercises 1–2, find the next shape in the pattern.

1. 

2. 

In Exercises 3–5, find the next number in the pattern.

3. 1, 2, 3, 11, 12, 13, 21, 22, 23, ...

4. 1, 10, 100, 1000, ...

5. 25, 27, 23, 25, 21, 23, 19, 21, 17, ...

6. Mayra has 16 oranges. Her father eats 4 of them, then she shares the rest equally between herself and her two brothers. How many do they get each?

Round Up

Analyzing problems is an important part of math and an important thing in real life. Ask yourself what is going on and what information you have, then try to combine the two. It can be difficult to put all the pieces together, but just take it slowly and don't panic.

Lesson 2.4.2

Important Information

California Standard:

Mathematical Reasoning 1.1

Analyze problems by identifying relationships, **distinguishing relevant from irrelevant information, identifying missing information**, sequencing and prioritizing information, and observing patterns.

What it means for you:

This Lesson is all about questions that don't contain the right amount of information. You'll practice identifying information that doesn't matter and information that is missing.

Key words:

- important
- missing
- information

Sometimes solving a problem might be hard because you have a lot of information, including some that you don't need. Other times, solving a problem might be hard (or even impossible) because you don't have enough information. This Lesson is a chance to practice dealing with both kinds of problems.

Sometimes You Have Too Much Information

Sometimes a math problem may try to confuse you by giving **information that you don't need**. You need to be able to figure out what information is important and what isn't.

Example 1

The price list for items sold at the school Snack Bar is shown on the right.
Maria has $10 and buys a sandwich and a drink. How much change should she get?

Menu	
Sandwich	$3.55
Salad	$2.25
Drink	$1.35
French fries	$0.90

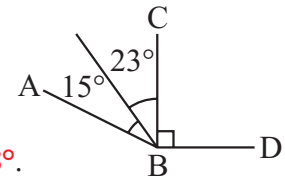
Solution

Maria has spent $3.55 + $1.35 = **$4.90**.
As she started with $10, she should have $10 – $4.90 = **$5.10 change**.

Although the question shows the price of salad and French fries, Maria doesn't buy either of those, so you have to ignore that information.

Example 2

What is the measure of the angle ABC?



Solution

The measure of angle ABC is 23° + 15° = **38°**.

The question also tells you that angle CBD is a right angle. But you don't need that information, so you can ignore it.

Guided Practice

In Exercises 1–3, identify any pieces of information that aren't needed.

1. Find the area of a rectangle which is sized 6 inches by 3 inches and has a perimeter of 18 inches.

2. Deidra earns $5 an hour as a babysitter and $6.50 as a lifeguard. Last weekend, she babysat for 4 hours. How much did she earn?

3. Mr. Smith's monthly bills are shown on the right. How much did he spend on rent and groceries?

Mr. Smith's Bills	
Rent	$700.00
Phone	$58.00
Utilities	$75.00
Groceries	$186.00

Don't forget:

Read the question carefully... Here, you just have to spot the unimportant pieces of information — you don't have to solve the problems themselves.

Some Problems Don't Give Enough Information

Sometimes you'll have a problem that you can't answer because you **haven't been told enough**. When that happens you need to be able to figure out what information is missing.

Example 3

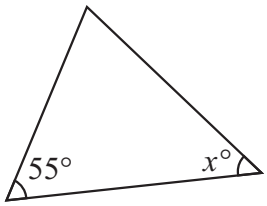
Raul is at the store to buy cheese, walnuts, and grapes. He wants to buy one pound of cheese and 1.5 pounds each of grapes and walnuts. If cheese costs $1.75 per pound, how much will Raul spend altogether?

Solution

You would need to know the **cost of walnuts** and the **cost of grapes** to answer this question.

Example 4

The local park has a triangular garden area, as shown on the right.



What is the value of x?

Solution

There is not enough information to solve this problem. You need to know the measures of **two** angles in a triangle to find the measure of the third angle. But the measure of only one angle is given.

Guided Practice

In Exercises 4–6, identify the piece of important information not given.

4. Mr. Mendoza is building a rectangular play yard for his children. The length of the play yard is 8 meters. What is the area?

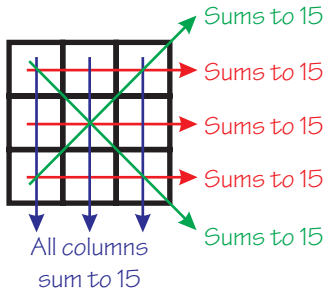
5. Ronald earned $175.00 working at the grocery store last week. How much does he earn per hour working at the grocery store?

6. Bill weighs 10 pounds more than his sister. What does Bill weigh?

It Can Be Hard to Tell If You Have Enough Information

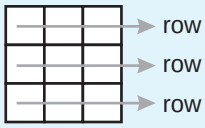
Sometimes it's not obvious if you have enough information or not. You have to start a problem and see where you can get to.

These next examples are about "magic squares." A magic square is a 3 × 3 grid containing the numbers **1 to 9**, where each row, column, and diagonal **adds up to the same number** (in these examples, 15).

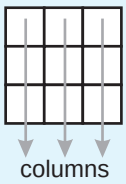


Don't forget:

Rows go across (they're horizontal).



Columns go down (they're vertical).



Example 5

Complete the magic square shown.
All the rows, columns, and diagonals must sum to 15.

	9	2
	5	

Solution

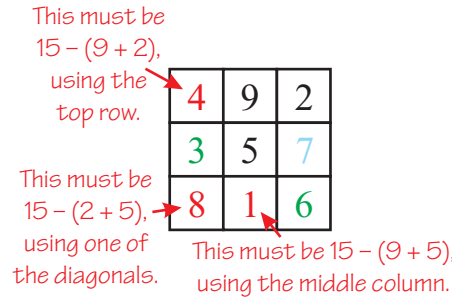
The numbers 2, 5, and 9 have already been used.
That means you have to put the numbers 1, 3, 4, 6, 7, and 8 in the grid, making sure that all rows, columns, and diagonals sum to 15.

There are a lot of gaps. At this stage, you can't be 100% sure you have enough information. But you can make a start...

First of all, you can fill in the numbers shown in red.

But then you have enough information to fill in the numbers in green.

And finally, you can fill in the number in the center-right box. It must be 7.



So here, you did have enough information.

Example 6

Do you have enough information to complete this magic square? All the rows, columns, and diagonals must sum to 15.

	9	
	5	3

Solution

You can work out the numbers in red straightaway.
But then there's no easy way to carry on.
It looks like you don't have enough information.

	9	
7	5	3
	1	

But... if you try to put the next number in, you discover something.
You can't put "8" in any of the colored squares.

- If you put it in the **green** square, then the top row sums to more than 15.
- If you put it in the **blue** square, then the top row also sums to more than 15.
- If you put it in the **red** square, then the left-hand column will sum to more than 15 (you can't use 0).

	9	
7	5	3
	1	

So "8" has to go in the bottom-right square.
And then you can fill in all the others.

2	9	4
7	5	3
6	1	8

So you did have enough information, but you had to think hard about it.

Guided Practice

7. Do you have enough information to complete the magic squares on the right? All the rows, columns, and diagonals must sum to 15.
What do you discover when you try to complete the second magic square?

		2
	5	9

	5	9
	4	

Independent Practice

In Exercises 1–6, say whether you have too much or too little information to answer the question. If you have too much information, then say which information you don't need. If you have too little, then say what information you would need.

1. At 4:00 p.m., the temperature was 45° Fahrenheit. Each hour, the temperature dropped by the same amount. What was the temperature at 7:00 p.m.?

2. Damian is ten minutes late for school. He set off at 8:30 a.m. and it took him 40 minutes to get to school. During that time he had a 5-minute stop to buy a drink. What time does school start?

3. Marissa is a doctor. On Tuesday, she sees 12 patients in the morning, and in the afternoon she sees a further 10. To help out another doctor, Jorge, she agrees to see 3 of his patients on Wednesday. How many patients did she see in total on Tuesday?

4. The Coleman family goes to the movies. Adult tickets sell for $4.50 and children's tickets sell for $2.00. How much will the Coleman family pay for tickets?

5. Mario needs to buy 120 meters of pipe, which he intends to divide into 8 equal parts. Pipe costs $10/meter. How much money does he need to buy the pipe?

6. Joshua is playing a game of tic-tac-toe against Luis. Luis says that for every game he loses, he will buy Joshua a soda. In the end, Luis loses four games. How much money must he spend on soda?

7. Do you have enough information to complete the magic squares on the right? All the rows, columns, and diagonals must sum to 15.

	5	
	9	4

	4	9
	5	

Now try these:
Lesson 2.4.2 additional questions — p437

Round Up

Being able to tell the difference between information that matters and information that doesn't matter is important in math and in real life. The main thing is not to panic. Just go through each fact and decide whether it is important or not. When you get to the end, see if you have enough information to answer the question.

Lesson 2.4.3

Breaking Up a Problem

This Lesson is about some very general ideas that you can use in all the other math lessons you do this year, next year, the year after that, and so on. The idea is to make the most of every little bit of math knowledge you have — a very useful thing to be able to do.

Don't Do More Work Than You Need to

Sometimes solving a problem **directly** is very difficult.
In those cases, look for any clever **shortcuts** you can to make things easier.

Example 1

Linda is working on an extra-credit assignment.
Her teacher wants her to find the last digit of 3^{20} without using a calculator. Linda is, however, allowed to use the table below.

How can Linda find the answer most efficiently?

n	3^n	Last Digit of 3^n
1	3	3
2	9	9
3	27	7
4	81	1
5	243	3

n	3^n	Last Digit of 3^n
6	729	9
7	2187	7
8	6561	1
9	19,683	3
10	59,049	9

Solution

One way would be to work out more and more powers of 3.
So she could work out 3^{11} by calculating $59{,}049 \times 3$.
Then she could find 3^{12} by multiplying this answer by 3, and so on until she got to 3^{20}. But it would take a long time, and it would be very easy to make a mistake.

However, Linda spots a **pattern** in the final digits.
They go 3, 9, 7, 1, and then repeat.

So by writing down 3, 9, 7, 1 again and again until she reaches the 20th number, Linda works out that the last digit of 3^{20} must be **1**.

Guided Practice

1. By making a table for small exponents, find the last digit of 4^{30}.

2. Marissa is asked to add up every number from 1 to 100. To try to find a pattern, she paired up the numbers in the sum. She added the first number (1) to the last (100), the second number (2) to the second to last (99), and so on. She found that each pair added up to 101. She then realized that she could find the answer by doing just one multiplication. What multiplication should she do?

California Standards:

Mathematical Reasoning 1.3
Determine when and how to break a problem into simpler parts.

Mathematical Reasoning 2.2
Apply strategies and results from simpler problems to more complex problems.

Mathematical Reasoning 2.4
Use a variety of methods, such as words, numbers, symbols, charts, graphs, **tables, diagrams,** and models, **to explain mathematical reasoning.**

Mathematical Reasoning 2.5
Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.

Mathematical Reasoning 2.7
Make precise calculations and check the validity of the results from the context of the problem.

What it means for you:

This is about being smart.

Key words

- question technique
- units
- reasonable

Check it out:

Write the numbers in rows of 4 until you've got 20 of them.

3, 9, 7, 1
3, 9, 7, 1
3, 9, 7, 1
3, 9, 7, 1
3, 9, 7, (1)

Don't forget:

3^2 means the same thing as 3×3. 3^3 means the same thing as $3 \times 3 \times 3$, and so on. The small raised number is called the exponent.

The next example has been broken up into several stages.
It's not the problem itself that's the important thing here, but the **approach to completing the task**.

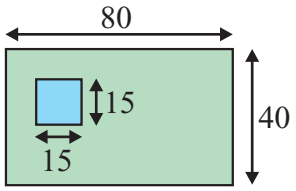
Break Problems Up into Smaller Parts

Sometimes a big, complicated problem is easier if you **split it up**.
So you take your **big**, **complicated** problem and break it up into many **smaller**, **simpler** problems that you can solve.

Don't forget:
There will often be different ways to break the problem up. Try to look for the way that will make things easiest for you.

Example 2

The diagram below shows a backyard with a swimming pool.
What area of the backyard is grass?



All lengths are in yards.

Solution — Part 1

You've actually seen a similar kind of problem already — in Section 2.3. What you did then was to break the big, complicated problem up into smaller, easier-to-solve problems. That's all you're doing here too.

To find the area of the grass, you can:
- Find the total area of the backyard.
- Find the area of the swimming pool.
- Subtract the area of the swimming pool from the area of the backyard.

Always Show How You're Solving the Problem

When you solve a problem in math, you should always **explain how** you got your answer. Use **any** method that explains your reasoning — for example words, numbers, symbols, charts, tables, graphs...

Check it out:
Doing this will help your teacher follow your work. But it will also help you when you check your work.

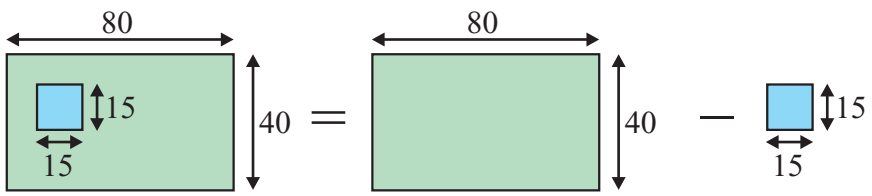
Example 2 (continued)

Look again at the problem in Example 2 above.

Solution — Part 2

You should always write down exactly how you're going to find your answer. So here, you could write:
Area of grass = Total area of backyard – Area of swimming pool

Or you could draw a diagram:



Always Show Your Work

Even if you use a **calculator** to do the actual calculations, it's best to write down all the different answers you found on the way.

Check it out:
If you write all your intermediate answers down, then it's easier to spot mistakes when you check your work.

Example 2 (continued)

Look again at the problem in Example 2 on the previous page.

Solution — Part 3

Total area of backyard = $80 \times 40 = 3200$
Area of swimming pool = $15 \times 15 = 225$

So area of grass = $3200 - 225 =$ **2975**

Write Your Answer Clearly, and Remember Units

There's still something **missing** from the above answer.
As it stands, it's not clear if the answer is 2975 m^2, 2975 in^2, 2975 yd^2, or something entirely different.

Example 2 (continued)

Look again at the problem in Example 2 on the previous page.

Solution — Part 4

Always check the question to see if you should include units.

Area of grass = $3200 - 225 =$ **2975 yd^2.**

Check Your Answers

You've found an answer. But you should always **check** it — it's easy to press the wrong button on a calculator, for example.

Example 2 (continued)

Look again at the problem in Example 2 on the previous page.

Solution — Part 5

Suppose your calculator gave you an answer of 31,775 yd^2.
You can quickly check if it's reasonable.

The area of grass has to be less than the total area of the backyard. The total area of the backyard was 80×40 $yd^2 = 3200$ yd^2, so the answer has to be less than this.

So the answer of 31,775 must be wrong.
It looks like you pressed the calculator's zero button too many times, giving 32,000 – 225 instead of 3200 – 225.

Don't forget:
You can use rounding to check your answer — see Section 1.4.

Guided Practice

3. Tyler and Ashlee both tried to multiply 1.3 by 2 and both got the wrong answer. Their work is shown below.
Identify the error each made, if possible.
Tyler: $1.3 \times 2 = 9.3$
Ashlee: $1.3 \times 2 = 2 \times (1 + 0.3) = (2 \times 1) + (2 \times 0.3) = 2 + 0.5 = 2.5$

4. At the amusement park, the roller coaster ride takes 2 minutes and the merry-go-round takes 3.5 minutes. Gabriela rode the roller coaster two times and the merry-go-round three times.
What is the total amount of time that she spent on rides?

5. A half-gallon container of orange juice sells for \$3.75, and a half-gallon container of apple juice sells for \$2.85.
Mr. Gonzales buys three containers of orange juice, and Ms. Chung buys two containers of apple juice.
How much more than Ms. Chung did Mr. Gonzales spend?

Solve the equations given in Exercises 6–9 and check your answer.

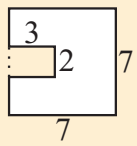
6. $y \div 3 = -16$ **7.** $r \times 9 = 72$
8. $p \times 11 = -66$ **9.** $q \div 6 = 6$

Check it out:
Check your answers to Exercises 6–9 by substituting your answer back into the original equation.

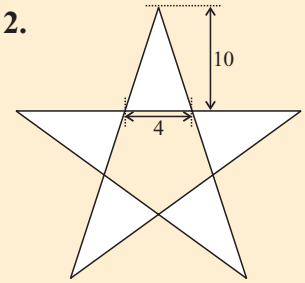
Independent Practice

Find the white shaded areas of the shapes given in Exercises 1–3.
All measurements are given in inches.

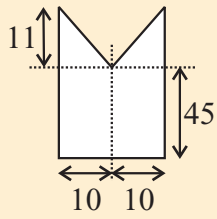
1.



2.



3.



4. By making a table for small exponents, find the last digit of 7^{20}.

5. Jim is selling necklaces at \$1 per necklace. Erin buys 4 and Olivia buys 6. Madison buys half of Erin's for \$1.50 per necklace.
How much more did Olivia spend than Madison?

6. Manuel is trying to find out if 174 467 093 477 549 819 816 can be divided exactly by 4. He notices that 12, 112, 212, 312, and 412 can all be divided exactly by 4. What pattern has he noticed? Find a pattern to help decide whether the large number can be divided exactly by 4.

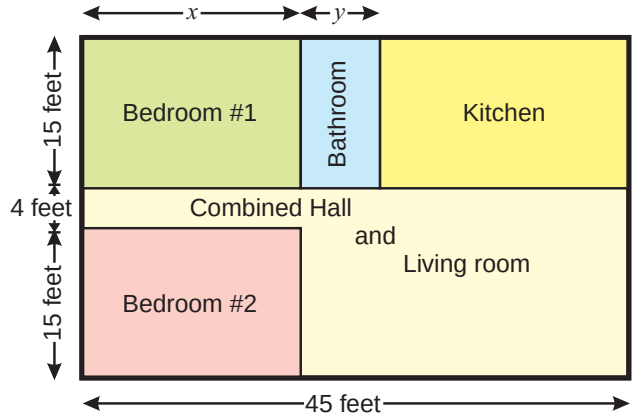
Now try these:
Lesson 2.4.3 additional questions — p438

Round Up

So you see... most of this is common sense. Always show your work, check your answers, and so on. But it's small things like this that really help you to get more questions right, more of the time, and that can't be a bad thing.

Chapter 2 Investigation
Design a House

You're an architect. Your clients know the layout they want in their new home, but they haven't yet decided on the exact measurements of all the rooms. So some variables have been used in the floorplan below.



Part 1:

Write an expression for the **area** of each of the 5 rooms in the house.

Things to think about:

- To check your answers, think about what these areas should **add up** to.

Part 2:

On the right is a price list for **carpets** and **floor tiles**.

The **bedrooms** and **hall/living room** are to be **carpeted**, but the **bathroom** and **kitchen** will be **tiled**.

Write an expression for the **total cost** of carpets and floor tiles.

Carpet: $2 per square foot
Tiles: $3 per square foot

Start by writing an expression for each room.

Extension

1) Write an expression for the **perimeter** of each room.
2) If the height of all the rooms is **9 feet**, and all the house's windows and doors have a total area of 162 ft^2, write an expression for the total internal wall area.
3) All the internal walls are to be painted, and paint costs $0.20 for each square foot of wall. How much will the paint cost for all the internal walls?
4) The clients decide that the floor area of the bedrooms is to be **270 square feet** each. Use this information to **find *x***. Can you find the final costs of paint and floor coverings?

Open-ended Extension

Explore **different floor layouts** for a house. Start with a **one-room** house.
You want your house to have a floor area of **400 square feet**. However, you want to do as little building as possible, so you want the total length of walls to be **as small as possible**.

1) What shape should your house be?
2) Would you get a different answer if the walls didn't have to be straight?
3) What if you want your house to have **three rooms**, each with an area of 400 square feet? What is the best layout if you want the total length of walls to be as short as possible?

Round Up

Remember... If you don't know the value of something, you can just give it a variable name instead. You might be able to work it out later, when you've found out some more information.

Chapter 3

Fractions and Percentages

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Lesson 3.1.1

Section 3.1
Understanding Fractions

You've worked with fractions in earlier grades and so you should already have a good idea of what they represent. This Lesson reviews some of those concepts and introduces the idea of negative fractions.

California Standard:

Number Sense 1.1

Compare and order positive and negative fractions, decimals, and mixed numbers **and place them on a number line.**

What it means for you:

You'll first review what fractions show. Then you'll mark positive and negative fractions that fall between –1 and +1 on a number line.

Key words:

- denominator
- equivalent fraction
- numerator
- negative

Don't forget:

You can think of the number on the bottom as how many parts there are in all. Then the number on the top is how many of those parts you are talking about.

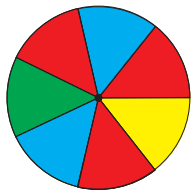
Fractions Can Show a Part of the Whole

A fraction can represent a part of a single object. For example:

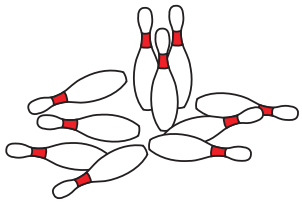
The spinner has been divided into seven equal parts (sevenths), and one of these parts is green.

$\frac{1}{7}$ (one-seventh) of a spinner is green.

$\frac{3}{7}$ (three-sevenths) of the spinner is red.



A fraction can also represent a certain number of a set of objects. For example:

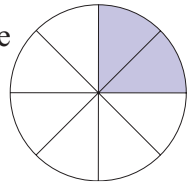


$\frac{3}{10}$ (three-tenths) of a set of ten bowling pins is still standing.

$\frac{7}{10}$ (seven-tenths) of the set has been knocked down.

Fractions Can Be Expressed in Different Ways

$\frac{2}{8}$ of this circle is shaded. You can see that this is the same as one quarter ($\frac{1}{4}$). $\frac{2}{8}$ and $\frac{1}{4}$ are **equivalent fractions**. Equivalent fractions have the same value.



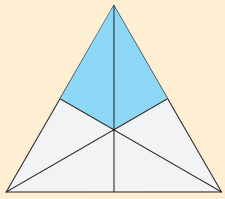
If you have 30 students in your math class and 15 are boys, you could say that $\frac{15}{30}$ of the students are boys. However, you could also say that $\frac{1}{2}$ of the students are boys. $\frac{15}{30}$ and $\frac{1}{2}$ are **equivalent fractions**.

Guided Practice

1. What fraction of the spinner above is blue?

2. 17 out of the 24 students in a class have a pet dog. Express the number that have a dog as a fraction.

3. What fraction of this triangle is blue? Give a fraction that is equivalent to this.



4. You invite 8 friends to a party, but two of them can't come. What fraction of your 8 friends can't come? Give a fraction that is equivalent to this.

Don't forget:

The numerator is on the top of the fraction.
The denominator is on the bottom.

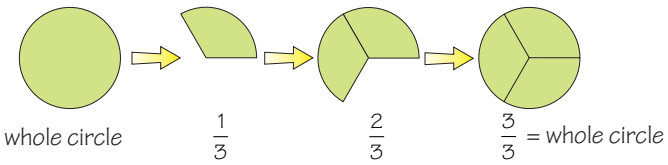
$\frac{3}{4}$ ⇦ numerator
⇦ denominator

Two-Halves and Three-Thirds Both Equal 1 Whole

If the numerator and the denominator of a fraction are the same, the fraction is equal to 1.

So, for example: $\frac{2}{2}=\frac{3}{3}=\frac{6}{6}=\frac{10}{10}=\frac{54}{54}=1$

The circle below is divided into thirds. Each segment is $\frac{1}{3}$ of the whole circle. Two segments make up $\frac{2}{3}$ of the whole circle, and all three of the segments make $\frac{3}{3}$ of the circle — which is equivalent to the **whole circle**.



Fractions Can Be Shown on a Number Line

Just like integers, fractions can be shown at positions on the number line. Think of 1 on the number line as representing one whole.

Positive fractions with a numerator smaller than the denominator, like $\frac{1}{2}$ or $\frac{4}{5}$, represent parts of a whole. This means they are **between 0 and 1** on the number line.

Example 1

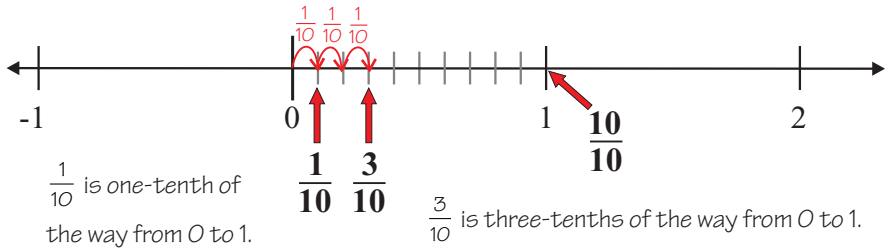
Show $\frac{1}{10}$ and $\frac{3}{10}$ on a number line.

Solution

The number 1 on the number line represents 1 whole.

It could also be written as $\frac{10}{10}$.

So divide the distance between 0 and 1 into 10 equal parts (tenths).



Guided Practice

Mark the fractions in Exercises 5–9 on a number line.

5. $\frac{1}{4}$ **6.** $\frac{1}{5}$ **7.** $\frac{1}{8}$ **8.** $\frac{2}{10}$ **9.** $\frac{8}{10}$

10. Which of the two fractions above are at exactly the same position?

Fractions Can Be Negative Too

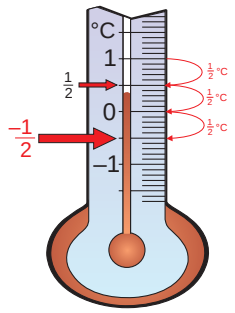
Imagine at 9 p.m. the temperature is 1 °C.

It then falls by $\frac{1}{2}$ a degree every hour until midnight.

At 10 p.m. it is $\frac{1}{2}$ °C, then at 11 p.m. it is at 0 °C.

Over the next hour it falls by another $\frac{1}{2}$ degree.

This takes it to $-\frac{1}{2}$ °C at midnight.



This illustrates that just like integers, fractions can be **negative**.

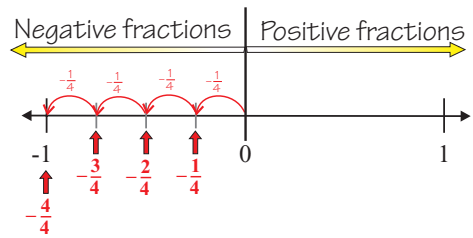
So you can have values such as $-\frac{1}{2}$, $-\frac{4}{5}$, or $-\frac{51}{73}$.

Negative Fractions Can Be Shown on the Number Line

Negative fractions are hard to picture in the same way as positive fractions. They can be shown on the number line though. Like negative integers, they are positioned to the **left** of zero.

A negative fraction with a numerator smaller than the denominator will always be **between 0 and –1** on the number line.

Part of this number line has been divided into fourths. $-\frac{1}{4}$, $-\frac{2}{4}$, and $-\frac{3}{4}$ are all between 0 and –1.



Don't forget:
There's lots more information about positive and negative integers in Section 1.1.

Negative integers **decrease in value** the further to the left you go. The same is true of negative fractions.

So the further to the right on the number line a negative fraction lies, the **greater** it is.

On the number line above, you can see that $-\frac{1}{4}$ is greater than $-\frac{3}{4}$, as it lies further to the right. You can write $-\frac{1}{4} > -\frac{3}{4}$, or $-\frac{3}{4} < -\frac{1}{4}$.

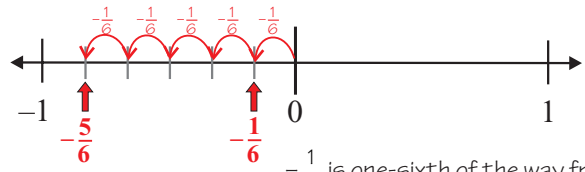
Example 2

Show $-\frac{1}{6}$ and $-\frac{5}{6}$ on a number line. Which is greater?

Solution

Divide the distance between 0 and –1 into 6 equal parts (sixths).

$-\frac{5}{6}$ is five-sixths of the way from 0 to –1.



$-\frac{1}{6}$ is one-sixth of the way from 0 to –1.

$-\frac{1}{6}$ lies further to the right, so it is greater, or $-\frac{1}{6} > -\frac{5}{6}$.

Guided Practice

Mark each pair of fractions in Exercises 11–13 on a number line. Say which of each pair is greater.

11. $-\frac{4}{5}$, $-\frac{1}{5}$ **12.** $-\frac{1}{8}$, $-\frac{5}{8}$ **13.** $-\frac{1}{10}$, $-\frac{7}{10}$

14. What do you notice about the positions on the number line of the following pairs of fractions: $-\frac{1}{4}$ and $\frac{1}{4}$, $-\frac{4}{5}$ and $\frac{4}{5}$?

Check it out:
Absolute value is the size of a number, ignoring any minus signs. For example, the absolute value of 4 is 4, but the absolute value of –4 is also 4. The absolute value of both 0.5 and –0.5 is 0.5.

Negative Fractions Behave Like Negative Integers

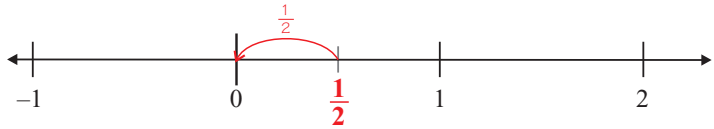
When you add together a positive fraction and a negative fraction with the same absolute value, you get back to zero.

Example 3

Use a number line to find $\frac{1}{2} + (-\frac{1}{2})$.

Solution

Start at $\frac{1}{2}$. When you added a negative integer you moved to the left. You add a negative fraction in the same way. So move $\frac{1}{2}$ a space to the left.



The solution is **0**.

Guided Practice

Use a number line to find the values of the following:

15. $-\frac{4}{5} + \frac{4}{5}$ **16.** $-\frac{1}{8} + \frac{1}{8}$ **17.** $\frac{7}{10} + (-\frac{7}{10})$

Independent Practice

1. Draw a rectangle. Shade and label $\frac{3}{5}$ of it.

2. Tim has visited 13 out of the 50 states. Express this as a fraction. What fraction of the states has he not visited?

3. 26 out of 52 playing cards are black. Express this as a fraction. What simpler fraction is this equal to?

Mark the fractions in Exercises 4–6 on number lines.

4. $\frac{2}{3}$ and $-\frac{2}{3}$ **5.** $\frac{2}{5}$ and $-\frac{2}{5}$ **6.** $\frac{1}{9}$ and $-\frac{1}{9}$

Now try these:
Lesson 3.1.1 additional questions — p438

Round Up

The fractions you've looked at in this Lesson are mainly proper fractions — where the numerator is smaller than the denominator. In the next Lesson you'll look at improper fractions. In these, the numerator is bigger than (or equal to) the denominator.

Lesson 3.1.2

Improper Fractions

California Standard:

Number Sense 1.1

Compare and **order positive and negative fractions**, decimals, **and mixed numbers and place them on a number line.**

What it means for you:

You'll develop techniques for dealing with improper fractions. This will give you the skills needed to place such fractions on the number line.

Key words:

- denominator
- improper fraction
- numerator
- proper fraction
- mixed number

Don't forget:

You met fractions in which the numerator is equal to the denominator in the last Lesson (like $\frac{6}{6}$) — they are always equivalent to 1.

In the last Lesson, the positive fractions you looked at mainly lay between 0 and +1 on the number line — they had numerators that were smaller than their denominators. In this Lesson, you'll look at improper fractions. In these, the numerator is bigger than or equal to the denominator.

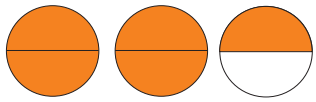
Improper Fractions — They're "Top-Heavy"

An improper fraction is one whose numerator is greater than or equal to the denominator.

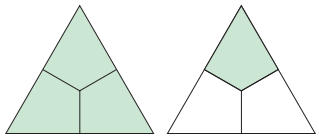
This is only true for positive numbers.

Some examples of improper fractions are shown below:

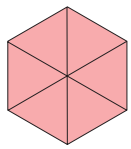
$\frac{5}{2}$ (five-halves) circles are shaded.



$\frac{4}{3}$ (four-thirds) triangles are shaded.



$\frac{6}{6}$ (six-sixths) of a hexagon is shaded.



Mixed Numbers — Whole Numbers Plus Fractions

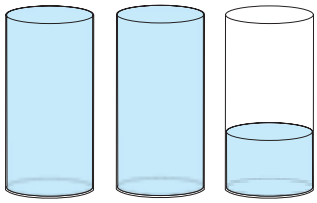
A mixed number is one that shows the sum of a whole number and a fraction.

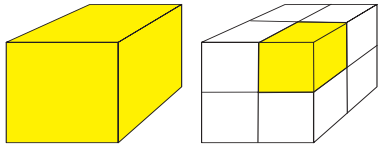
Here are some examples of mixed numbers:

$3\frac{1}{4}$ (three and one-quarter) squares are shaded.



There are $2\frac{1}{3}$ (two and one-third) glasses of water.





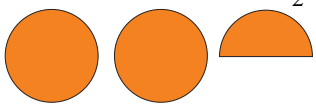
$1\frac{1}{8}$ (one and one-eighth) large cubes are shaded.

Mixed Numbers Have Equivalent Improper Fractions

The diagrams shown on the previous page can be represented by **both** mixed numbers and improper fractions.

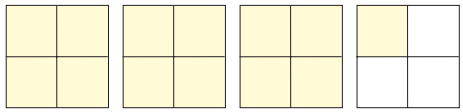
For example:

$\frac{5}{2}$ shaded circles is the same as saying $2\frac{1}{2}$ shaded circles.



2 whole circles and one half-circle are shaded.

$3\frac{1}{4}$ squares are shaded is the same as saying $\frac{13}{4}$ squares are shaded.

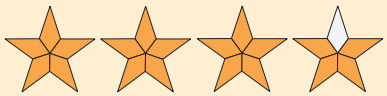


Thirteen quarter-squares are shaded.

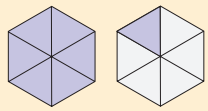
Guided Practice

Represent the shaded fraction of these groups of shapes using a mixed number and an improper fraction.

1.

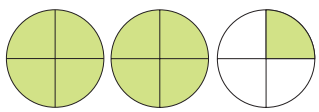


2.



Converting Mixed Numbers to Improper Fractions

Suppose you need to convert $2\frac{1}{4}$ to an improper fraction. You could draw a diagram. The mixed number $2\frac{1}{4}$ can be represented by this diagram:



The diagram shows that $2\frac{1}{4}$ consists of a total of **9 quarters**, or $\frac{9}{4}$.

But you don't really need a diagram — you can work out the numerator and the denominator without one.

The new **denominator** in the improper fraction is always the same as in the equivalent mixed number.

A quick way of calculating the new **numerator** is to **multiply the whole number part by the denominator and add the old numerator**.

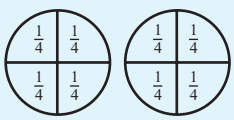
Mixed number $\rightarrow 2\frac{1}{4} = \frac{(2\times 4)+1}{4} = \frac{9}{4} \leftarrow$ Improper fraction

the denominator remains the same.

Check it out:

In this method, you're working out the number of fourths in $2\frac{1}{4}$.

There are 2 × 4 = 8 fourths in two.



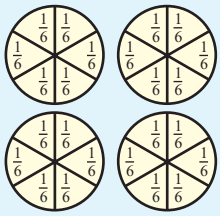
Then there is also one extra fourth that was expressed as the fraction part.



8 fourths, plus 1 fourth equals 9 fourths, also written as the improper fraction, $\frac{9}{4}$.

Don't forget:

There are 4 × 6 = 24 sixths in four.



Then there are also an extra five sixths that were expressed as a fraction before.



So there are (4 × 6) + 5 = 29 sixths in total.

Example 1

Write $4\frac{5}{6}$ as an improper fraction.

Solution

$$4\frac{5}{6} = \frac{(4\times 6)+5}{6}$$

$$= \frac{29}{6}$$

To find the improper fraction's numerator, multiply the whole number part by the denominator, then add the original numerator to this.

Keep the denominator the same.

$\frac{29}{6}$ is an improper fraction equal to $4\frac{5}{6}$.

Guided Practice

Write the following mixed numbers as improper fractions.

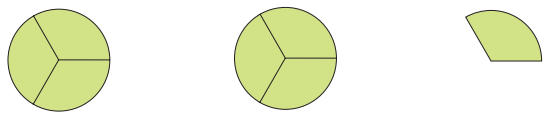
3. $5\frac{1}{4}$ **4.** $4\frac{3}{5}$ **5.** $1\frac{3}{8}$ **6.** $10\frac{1}{10}$

7. $8\frac{2}{3}$ **8.** $7\frac{4}{5}$ **9.** $2\frac{3}{7}$ **10.** $9\frac{7}{8}$

Converting Improper Fractions to Mixed Numbers

Now suppose you need to convert $\frac{7}{3}$ to a mixed number. The improper fraction $\frac{7}{3}$ is seven-thirds. But to make a whole, you only need three-thirds. Seven-thirds can be split up like this:

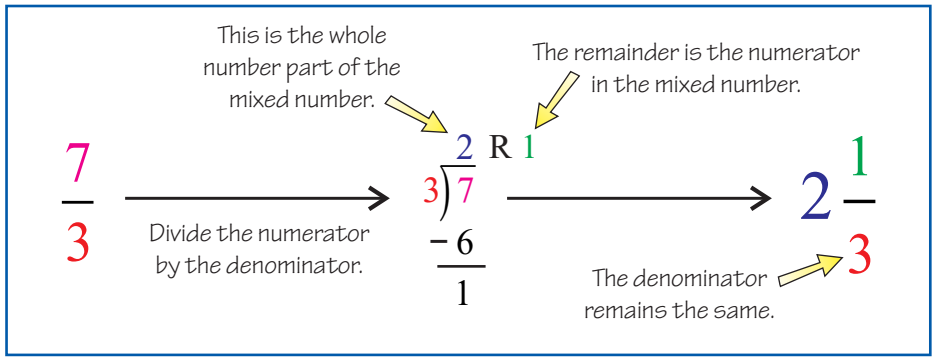
three-thirds + three-thirds + one-third = seven-thirds



From the diagram, you can see that $\frac{7}{3}$ is the same as $2\frac{1}{3}$.

A faster way of doing this is by **division**. To get the **whole number part**, you need to find out how many times the denominator of the improper fraction is contained in the numerator of the improper fraction.

The **remainder** (the number of parts left over) becomes the numerator of the fraction in the mixed number.



Example 2

Write $\frac{59}{4}$ as a mixed number.

Solution

$\frac{59}{4} \rightarrow 4\overline{)59}$ $\begin{array}{r} 14 \text{ R } 3 \\ -\ 4 \\ \hline 19 \\ -\ 16 \\ \hline 3 \end{array}$

Divide the numerator by the denominator.

$\rightarrow 14\frac{3}{4}$

$\frac{59}{4}$ is written as $14\frac{3}{4}$ as a mixed number.

Guided Practice

Write the following improper fractions as mixed numbers.

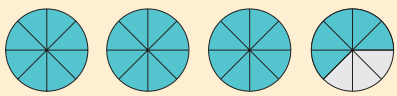
11. $\frac{8}{5}$ **12.** $\frac{9}{4}$ **13.** $\frac{11}{8}$ **14.** $\frac{12}{11}$

15. $\frac{16}{13}$ **16.** $\frac{21}{2}$ **17.** $\frac{23}{7}$ **18.** $\frac{91}{8}$

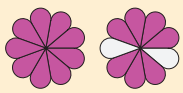
Independent Practice

Represent the shaded fraction of the groups of shapes in Exercises 1–2 by both a mixed number and an improper fraction.

1.



2.



Convert the mixed numbers in Exercises 3–10 to improper fractions.

3. $2\frac{5}{7}$ **4.** $4\frac{1}{6}$ **5.** $1\frac{7}{11}$ **6.** $8\frac{9}{13}$

7. $27\frac{1}{2}$ **8.** $12\frac{6}{7}$ **9.** $19\frac{2}{3}$ **10.** $6\frac{11}{14}$

Convert the improper fractions in Exercises 11–18 to mixed numbers.

11. $\frac{87}{43}$ **12.** $\frac{12}{7}$ **13.** $\frac{37}{8}$ **14.** $\frac{53}{4}$

15. $\frac{53}{6}$ **16.** $\frac{61}{9}$ **17.** $\frac{30}{7}$ **18.** $\frac{72}{5}$

Now try these:

Lesson 3.1.2 additional questions — p438

Round Up

After this Lesson, you should be able to convert improper fractions to mixed numbers.
This is important for placing fractions on the number line — you'll see this in the next Lesson.
You should also be able to convert mixed numbers to improper fractions.
You'll use this skill in adding, subtracting, multiplying, and dividing fractions.

Lesson 3.1.3

More on Fractions

In this Lesson, you'll practice putting improper fractions and mixed numbers on the number line. You'll also look at negative improper fractions and place them on the number line.

California Standard:

Number Sense 1.1

Compare and order positive and negative fractions, decimals, **and mixed numbers and place them on a number line.**

What it means for you:

You'll mark positive and negative fractions on the number line using the techniques you practiced for dealing with improper fractions.

Key words:

- denominator
- improper fraction
- numerator
- proper fraction
- mixed number

Don't forget:

Numbers further to the right on the number line are greater.

For example: $2\frac{1}{4}$ is greater than 2 so it is placed further to the right.

Don't forget:

Convert an improper fraction to a mixed number by dividing.

$$\begin{array}{r} 3\text{ R}2 \\ 3\overline{)11} \\ \underline{-9} \\ 2 \end{array}$$

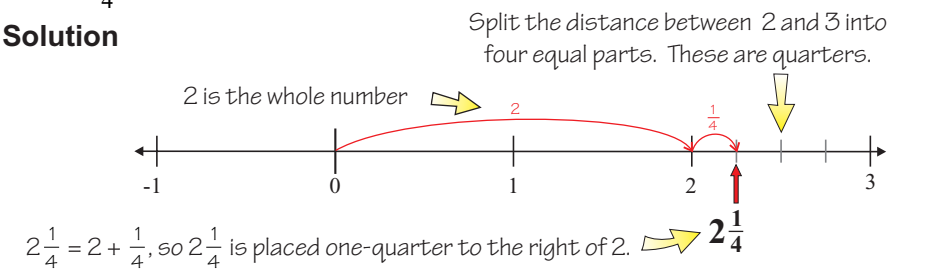
Placing Positive Mixed Numbers on a Number Line

When placing a positive mixed number on a number line, first find the position of the **whole number**. Then move the value of the **fraction** to the right. This is more easily shown with an example.

Example 1

Place $2\frac{1}{4}$ on the number line.

Solution



It's a good idea to check that your answer is sensible.

$2\frac{1}{4}$ is greater than 2 but less than 3. It is also closer to 2 than to 3. This matches the position marked on the number line.

Convert Improper Fractions to Mixed Numbers First

It's very difficult to place an improper fraction, like $\frac{11}{3}$ or $\frac{7}{4}$, on the number line. The best thing to do is convert the improper fraction to a **mixed number** and then place it on the number line.

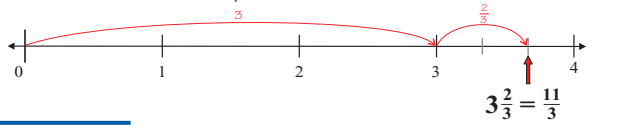
Example 2

Place $\frac{11}{3}$ on the number line.

Solution

$\frac{11}{3} = 3\frac{2}{3}$, so...

Convert to a mixed number, then place it on the number line and check.



Guided Practice

Place the following fractions on the number line.

1. $5\frac{1}{4}$
2. $4\frac{3}{5}$
3. $1\frac{3}{8}$
4. $\frac{8}{3}$
5. $\frac{7}{5}$
6. $\frac{23}{4}$

Placing Negative Mixed Numbers on a Number Line

Like all negative numbers, negative mixed numbers are placed to the **left** of zero on the number line.

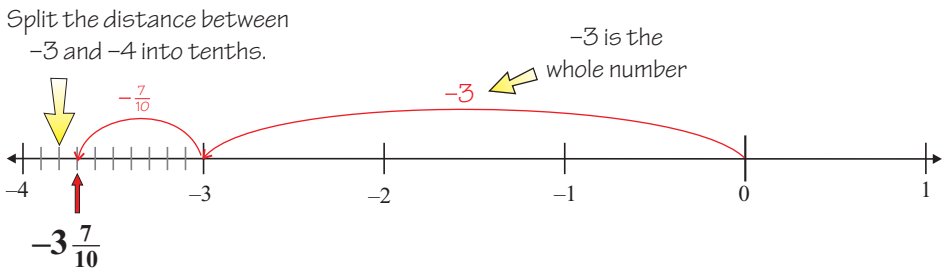
However, it's not always obvious exactly where they should be placed.

Example 3

Place $-3\frac{7}{10}$ on the number line.

Solution

$-3\frac{7}{10}$ lies the same distance to the left of zero as $3\frac{7}{10}$ lies to the right. So to place it on the number line, you do exactly as you would for a positive number, but to the left instead of the right.



Then a quick check: $-3\frac{7}{10}$ is between –3 and –4. It's closer to –4 than to –3. This matches the position marked on the number line.

Check it out:

$-3\frac{7}{10}$ can be thought of as being $\frac{7}{10}$ "more negative" than –3, or as $-3 + (-\frac{7}{10})$. So it's $\frac{7}{10}$ further left on the number line than –3.

Guided Practice

Place the following mixed numbers on the number line.

7. $-2\frac{1}{2}$ **8.** $-4\frac{1}{3}$ **9.** $-1\frac{3}{8}$

10. $-10\frac{1}{5}$ **11.** $-4\frac{3}{4}$ **12.** $-1\frac{4}{5}$

Change Negative Improper Fractions to Mixed Numbers

Just like with positive improper fractions, it is easiest to convert negative improper fractions to **mixed numbers** before you place them on the number line.

You convert negative improper fractions to mixed numbers in exactly the same way as positive improper fractions, but remember the **negative sign**. The example on the next page shows this.

Example 4

Write $-\frac{47}{3}$ as a mixed number and place it on a number line.

Solution

First find $\frac{47}{3}$ by dividing the numerator 47 by the denominator 3.

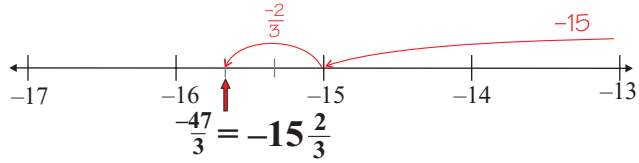
$$\frac{47}{3} \rightarrow 3\overline{)47} \rightarrow 15\frac{2}{3}$$

$$\begin{array}{r} 15 \text{ R } 2 \\ 3\overline{)47} \\ \underline{3\downarrow} \\ 17 \\ \underline{15} \\ 2 \end{array}$$

But remember the **negative sign**.

So $-\frac{47}{3}$ is $\mathbf{-15\frac{2}{3}}$ as a mixed number.

Put it on the number line as usual.



Check it out:

$-\frac{47}{3}$ can also be written as $\frac{47}{-3}$ or $\frac{-47}{3}$.

Guided Practice

Write the following improper fractions as mixed numbers and place them on a number line.

13. $-\frac{7}{4}$ **14.** $-\frac{39}{5}$ **15.** $-\frac{33}{8}$

16. $-\frac{17}{6}$ **17.** $-\frac{66}{10}$ **18.** $-\frac{21}{2}$

Independent Practice

Convert the improper fractions in Exercises 1–3 to mixed numbers.

1. $\frac{13}{5}$ **2.** $-\frac{27}{4}$ **3.** $-\frac{36}{7}$

State which one of each pair of fractions in Exercises 4–6 is greater.

4. $-4\frac{3}{7}$, $-4\frac{6}{7}$ **5.** $\frac{28}{6}$, $\frac{29}{6}$ **6.** $-12\frac{7}{8}$, $-\frac{101}{8}$

Show the fractions in Exercises 7–9 on a number line.

7. $18\frac{5}{6}$, $18\frac{1}{6}$ **8.** $-\frac{14}{3}$, $-\frac{13}{3}$ **9.** $-17\frac{5}{9}$, $-\frac{153}{9}$

Now try these

Lesson 3.1.3 additional questions — p439

Round Up

After this Lesson, you should be able to place any positive or negative fraction or mixed number on the number line. Placing fractions on the number line is one way of comparing fractions with different denominators, but in the next Lesson you'll learn a quicker method, involving converting them to decimals.

Lesson 3.1.4

Fractions and Decimals

In this Lesson, you'll learn how to convert fractions to their equivalent decimals. This allows you to decide whether a fraction or a decimal is greater.

California Standard:

Number Sense 1.1

Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.

What it means for you:

You'll learn how to compare a fraction with a decimal by first converting the fraction to a decimal.

Key words:

- decimal
- denominator
- equivalent
- numerator
- terminating decimal
- repeating decimal
- improper fraction

Don't forget:

1.3 means "1 plus 3 tenths" or $1\frac{3}{10}$. See Section 1.3 for more information.

Don't forget:

Make sure you do the division the correct way around. You are finding out how many times the bottom number goes into the top number.

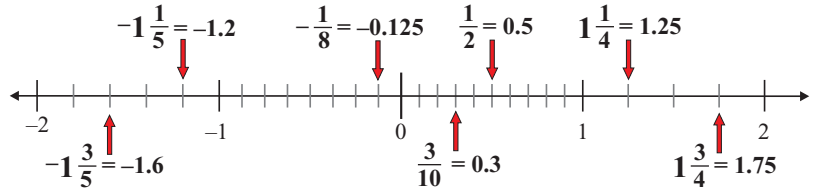
Don't forget:

Doing the division on a calculator will give the same answer.

Every Fraction Has an Equivalent Decimal

Every fraction has an **equivalent decimal**. A fraction and its equivalent decimal lie at the **exact** same point on the number line. Mixed numbers also have equivalent decimals.

For example:



Convert Fractions to Decimals by Division

Fractions represent **division**. You can write a fraction as a decimal by dividing the numerator by the denominator.

For example: $\frac{1}{2} = 1 \div 2 = 0.5$

$\frac{7}{4} = 7 \div 4 = 1.75$

Example 1

Write $\frac{5}{8}$ as a decimal.

Solution

Divide the numerator by the denominator.

$\frac{5}{8} \rightarrow 8\overline{)5.000}$ = 0.625

	8 goes into 50 six times...
− 48	6 × 8 = 48...
20	8 goes into 20 twice...
− 16	2 × 8 = 16...
40	8 goes into 40 five times...
− 40	5 × 8 = 40...
0	there's no remainder, so the division is finished.

$\frac{5}{8}$ is **0.625** as a decimal.

Convert Improper Fractions in the Same Way

This example shows that improper fractions are converted to decimals in the same way.

Don't forget:
When you convert an improper fraction to a decimal, you get a decimal greater than 1 or less than –1.

Example 2

Write $\frac{9}{5}$ as a decimal.

Solution

Divide the numerator by the denominator.

$\frac{9}{5} \rightarrow 5\overline{)9.0}$ — 1.8

-5 — 5 goes into 9 once...

40 — $1 \times 5 = 5$...

-40 — 5 goes into 40 eight times...

0 — $5 \times 8 = 40$...

there's no remainder, so the division is finished.

$\frac{9}{5}$ is **1.8** as a decimal.

Check it out:
To convert a mixed number to a decimal, just convert the fraction part and add it to the whole number part.
For example:
$2\frac{1}{2} = 2 + \frac{1}{2} = 2 + 0.5 = 2.5$

Guided Practice

Convert these fractions to decimals without using a calculator.

1. $\frac{1}{4}$ **2.** $\frac{1}{8}$ **3.** $\frac{8}{5}$ **4.** $\frac{11}{10}$

5. $2\frac{3}{4}$ **6.** $4\frac{7}{8}$ **7.** $9\frac{6}{15}$ **8.** $4\frac{6}{10}$

A Fraction Might Become a Repeating Decimal

Check it out:
When you divide 1 by 7 on your calculator, you see 0.1428571 on the screen. This doesn't look like a repeating decimal, but it is. The digits 142857 repeat over and over again, so you could write $\frac{1}{7} = 0.\overline{142857}$.

The fractions you have used so far this Lesson all have a **terminating decimal** equivalent. The decimals stop after a certain number of digits.

Other fractions have **repeating decimal** equivalents.
These go on forever, repeating the same digits.

For example: $\frac{1}{3} = 0.3333333...$, $\frac{1}{11} = 0.09090909...$

Repeating decimals are often written with a bar over the repeated digits.

So 0.333333... can be written as $0.\overline{3}$, and 0.09090909... as $0.\overline{09}$.

Check it out:
A fraction can be converted to either:
i) a terminating decimal
ii) a repeating decimal
There are no other possibilities.

Guided Practice

Convert these fractions into decimals using a calculator.

9. $\frac{2}{3}$ **10.** $\frac{7}{9}$ **11.** $\frac{6}{11}$ **12.** $\frac{5}{6}$

Comparing a Fraction with a Decimal

It is difficult to compare a fraction with a decimal directly. It's best to convert the fraction to a **decimal** and then compare the two decimals.

This might be necessary if you need to plot two numbers on a number line.

Check it out:

Sometimes you don't need to do a division to compare an improper fraction with a decimal. Converting the improper fraction to a mixed number may be enough. For example:

Which is greater: $\frac{13}{5}$ or 1.9?

$\frac{13}{5}$ is equivalent to $2\frac{3}{5}$.

You can easily see that this is greater than 1.9.

Example 3

Which is greater: $\frac{6}{5}$ or 1.15?

Solution

$\frac{6}{5} \rightarrow$

$$\begin{array}{r} 1.2 \\ 5\overline{)\,6.0} \\ -\,5\downarrow \\ \hline 1\,0 \\ -\,1\,0 \\ \hline 0 \end{array}$$

Convert $\frac{6}{5}$ into a decimal.

1.2 is greater than 1.15.

Compare the decimal produced with the decimal in the question.

Guided Practice

Which is the greater in each of these pairs?

13. $\frac{7}{8}$ or 0.85 **14.** $\frac{8}{5}$ or 1.7 **15.** $\frac{6}{32}$ or 0.2

Independent Practice

Convert the fractions in Exercises 1–4 to decimals without using a calculator.

1. $\frac{3}{5}$ **2.** $3\frac{7}{8}$ **3.** $\frac{3}{10}$ **4.** $2\frac{4}{5}$

Convert the fractions in Exercises 5–8 to decimals using a calculator.

5. $\frac{3}{11}$ **6.** $2\frac{3}{9}$ **7.** $3\frac{1}{6}$ **8.** $\frac{8}{11}$

Now try these:

Lesson 3.1.4 additional questions — p439

Put the sets of numbers in Exercises 9–10 in order, from least to greatest.

9. 4.32, $\frac{39}{8}$, $4\frac{4}{5}$ **10.** 6.7, $6\frac{5}{8}$, $6\frac{2}{3}$

11. Miranda needs a piece of pipe 6.81 meters long.

The store stocks the following sizes: $6\frac{3}{4}$ m, $6\frac{7}{8}$ m, $6\frac{13}{16}$ m.

Which size should she buy so that she has the least waste?

Round Up

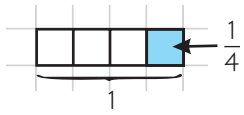
After this Lesson, you should be able to convert fractions to terminating decimals using long division. This is really handy if you're doing a calculation that involves both a fraction and a decimal.

Section 3.2 introduction — an exploration into:

Multiplying Fractions: an Area Model

Multiplying fractions has a lot in common with multiplying integers. For example, 3 × ½ means "3 groups of ½," in the same way that 3 × 4 means "3 groups of 4." You can show this using graph paper. In fact, using graph paper you can work out all kinds of fraction multiplications.

You can show fractions on graph paper. Here's **one fourth**, or $\frac{1}{4}$.



To **multiply** this, you need to make **groups**. You can use graph paper.

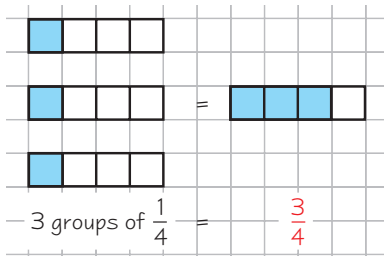
Example

Use graph paper to multiply $3 \times \frac{1}{4}$.

Solution

Draw and shade rectangles to show 3 groups of $\frac{1}{4}$.

Using the picture, you can see that $3 \times \frac{1}{4} = \frac{3}{4}$.



Exercises

1. Do these multiplications by drawing rectangles on graph paper to represent the fractions.

a. $2 \times \frac{1}{3}$ **b.** $3 \times \frac{2}{7}$

You can even multiply **two fractions** together using graph paper.

Example

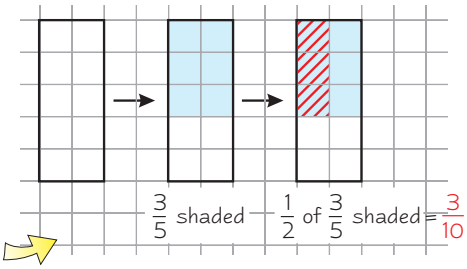
Use graph paper to multiply $\frac{1}{2} \times \frac{3}{5}$.

Solution

Draw a rectangle on graph paper that is **2 squares wide** by **5 squares tall**.

2 and 5 are the fractions' denominators.

Shade in **3 of the 5 rows** to show $\frac{3}{5}$ of the whole rectangle.



Now shade **1 of the 2 shaded columns** in a different color to show $\frac{1}{2}$ of $\frac{3}{5}$, or $\frac{1}{2} \times \frac{3}{5}$.

The part shaded both colors shows $\frac{1}{2} \times \frac{3}{5}$. You can see $\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$.

Exercises

2. Do these multiplications by drawing rectangles on graph paper to represent the fractions.

a. $\frac{2}{5} \times \frac{1}{3}$ **b.** $\frac{5}{6} \times \frac{3}{4}$

Round Up

Using graph paper like this is a good way to see what it actually means when you multiply fractions. In the next few Lessons, you'll learn a way to multiply fractions more quickly.

Lesson 3.2.1

Section 3.2
Multiplying Fractions by Integers

You've already practiced placing fractions in their correct positions on number lines. In this Lesson you'll use the number line to see how multiplying fractions by integers relates to multiplying two integers.

California Standard:

Number Sense 2.1

Solve problems involving addition, subtraction, **multiplication**, and division **of positive fractions** and explain why a particular operation was used for a given situation.

What it means for you:

You'll learn the skills needed to multiply fractions by positive and negative integers.

Key words:

- integer
- fraction
- numerator
- denominator
- proper fraction
- improper fraction
- mixed number

Don't forget:

Integers are numbers like:
0, 1, 2, 3, 4...,
or –1, –2, –3, –4...

Don't forget:

To convert an improper fraction to a mixed number, divide the top number by the bottom number using long division.

For example: $\frac{15}{4}$

$$4\overline{)15} \quad 3 \text{ R } 3$$
$$-12$$
$$3$$

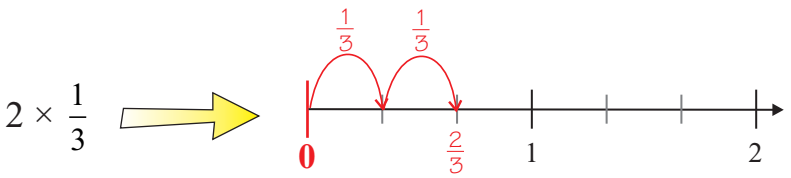
The remainder becomes the numerator of the fraction.

So $\frac{15}{4}$ is equivalent to $3\frac{3}{4}$.

Multiplying Fractions and Integers on the Number Line

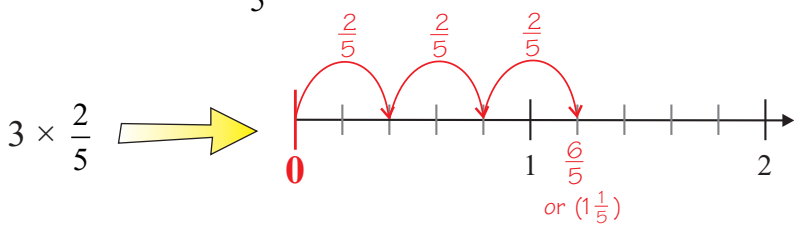
To multiply the fraction $\frac{1}{3}$ by the integer 2, think of it as "2 groups of $\frac{1}{3}$."

You can then model the calculation on a **number line**:



From the number line you can see that $2 \times \frac{1}{3} = \frac{2}{3}$.

In a similar way, $3 \times \frac{2}{5}$ can be modeled:



Again, using the number line you can see that $3 \times \frac{2}{5} = \frac{6}{5}$.

In each example, the **numerator** of the fraction has been multiplied by the **integer**, while the **denominator** has stayed the same.

To multiply a fraction by an integer:
multiply the numerator by the integer and keep the denominator the same.

Example 1

What is $5 \times \frac{3}{4}$?

Solution

Multiply the numerator 3 by 5, and keep the denominator as 4.

$$5 \times \frac{3}{4} = \frac{5 \times 3}{4}$$

$$= \frac{15}{4} \text{ or } 3\frac{3}{4}$$

Don't forget:

It doesn't matter in which order you multiply two values. You always get the same answer — this is the commutative property of multiplication.

So, $\frac{4}{5} \times 10 = 10 \times \frac{4}{5}$.

Example 2

What is $\frac{4}{5} \times 10$?

Solution

Multiply the numerator 4 by 10, and keep the denominator as 5.

$$\frac{4}{5} \times 10 = \frac{4 \times 10}{5}$$

$$= \frac{40}{5} = 8$$ **Simplify your answer**

5 divides into 40 exactly 8 times. There is no remainder, so $\frac{40}{5}$ is equivalent to an integer rather than a mixed number.

Guided Practice

Evaluate the following.

1. $6 \times \frac{1}{7}$ **2.** $\frac{2}{9} \times 4$ **3.** $12 \times \frac{1}{6}$ **4.** $15 \times \frac{2}{5}$

Multiply Fractions by Negatives in the Same Way

You saw in Chapter 1 that multiplying a **positive number** by a **negative number** gives a **negative** result. The same rule holds when one of the numbers is a fraction.

For example, take the following multiplication: $-5 \times \frac{3}{4}$

You've seen that to find $5 \times \frac{3}{4}$, you multiply the numerator of the fraction (3) by the integer (5), and keep the denominator the same.

This gives $5 \times \frac{3}{4} = \frac{15}{4} = 3\frac{3}{4}$.

Then using the "negative × positive = negative" rule,

$-5 \times \frac{3}{4} = -\frac{15}{4} = -3\frac{3}{4}$.

Multiplying a positive fraction by a negative integer gives a negative result.

Don't forget:
Use the same method to convert negative improper fractions to mixed numbers as you used for positive improper fractions — remember the minus sign, though.

Example 3

What is $-5 \times \frac{7}{8}$?

Solution

Multiply the numerator by 5, keep the denominator as 8, and remember the **negative sign**.

$$-5 \times \frac{7}{8} = -\frac{5 \times 7}{8}$$

$$= -\frac{35}{8} \text{ or } -4\frac{3}{8}$$

Guided Practice

Evaluate the following.

5. $-5 \times \frac{1}{6}$ **6.** $\frac{2}{9} \times -1$ **7.** $-8 \times \frac{2}{23}$ **8.** $\frac{5}{16} \times -3$

Evaluate the following.
Give your answer as an integer or mixed number.

9. $-12 \times \frac{2}{3}$ **10.** $-8 \times \frac{2}{5}$ **11.** $\frac{1}{6} \times -24$ **12.** $13 \times \frac{3}{8}$

Multiplying Mixed Numbers by Integers

Multiplying **mixed numbers** by integers is a very similar process — you just need to write the mixed number as an **improper fraction** before you multiply the numerator by the integer.

Example 4

Evaluate $4 \times 1\frac{2}{3}$.

Solution

$$4 \times 1\frac{2}{3} = 4 \times \frac{(1 \times 3) + 2}{3}$$ Convert mixed number to an improper fraction

$$= 4 \times \frac{5}{3}$$ Multiply integer by numerator

$$= \frac{20}{3} \text{ or } 6\frac{2}{3}$$

Multiplying mixed numbers by negative integers works in the same way.

Example 5

Evaluate $2\frac{3}{5} \times (-10)$.

Solution

$$2\tfrac{3}{5} \times (-10) = \frac{(2\times 5)+3}{5} \times (-10)$$ **Convert mixed number to an improper fraction**

$$= \frac{13}{5} \times (-10)$$ **Multiply integer by numerator**

$$= -\frac{130}{5}$$ **Simplify**

$$= -26$$

Don't forget:
5 divides into 130 exactly 26 times — there's no remainder, so $\frac{130}{5}$ simplifies to just 26.

Guided Practice

Evaluate the following.
Give your answers as mixed numbers or integers.

13. $-5 \times 2\frac{3}{4}$ **14.** $1\frac{1}{15} \times (-1)$ **15.** $3 \times 3\frac{3}{8}$ **16.** $4\frac{2}{3} \times (-3)$

Independent Practice

Multiply the fractions given in Exercises 1–4.

1. $3 \times \frac{3}{16}$ **2.** $\frac{2}{5} \times 2$ **3.** $-1 \times \frac{1}{6}$ **4.** $-5 \times \frac{3}{16}$

Evaluate the expressions given in Exercises 5–12.
Leave your answer as an integer or mixed number.

5. $10 \times \frac{1}{2}$ **6.** $\frac{4}{3} \times 3$ **7.** $-4 \times \frac{3}{8}$ **8.** $-40 \times \frac{5}{8}$

9. $-7 \times \frac{5}{33}$ **10.** $\frac{2}{5} \times 8$ **11.** $-13 \times \frac{4}{5}$ **12.** $-5 \times \frac{14}{67}$

Now try these:
Lesson 3.2.1 additional questions — p439

Evaluate the expressions given in Exercises 13–21.

13. $-2 \times 2\frac{1}{3}$ **14.** $1\frac{2}{5} \times 3$ **15.** $-1 \times 7\frac{1}{2}$

16. $-4 \times \frac{6}{5}$ **17.** $-1\frac{3}{4} \times 5$ **18.** $8 \times 1\frac{1}{11}$

19. $-2 \times 1\frac{5}{9}$ **20.** $3 \times 3\frac{1}{7}$ **21.** $7 \times 1\frac{3}{10}$

Round Up

So that's the basic method for multiplying fractions by integers. Many word problems that you'll come across will make you put this skill to use — there are some in the next Lesson.
It's also important for simplifying or solving algebraic equations.

Lesson 3.2.2

More on Multiplying Fractions by Integers

In this Lesson, you'll start by reviewing the skills learned last Lesson. Then you'll see how multiplication of fractions and integers can be used in real-life problems.

California Standards:

Number Sense 2.1

Solve problems involving addition, subtraction, **multiplication**, and division **of positive fractions and explain why a particular operation was used for a given situation.**

Number Sense 2.2

Explain the meaning of multiplication and division **of positive fractions and perform the calculations** (e.g., $\frac{5}{8} \div \frac{15}{16} = \frac{5}{8} \times \frac{16}{15} = \frac{2}{3}$).

What it means for you:

You'll review the skills needed to multiply fractions by integers. You'll also develop an understanding of how multiplication relates to problems involving fractions.

Key words:

- integer
- fraction
- denominator
- numerator
- mixed number

Don't forget:

When you multiply a fraction by an integer, the denominator stays the same.

Don't forget:

Make sure that you read what the question is asking. If it had asked, "How many packs will Pedro have to buy?", the answer would be 10 packs. This is because the store is unlikely to sell him $\frac{5}{8}$ of a pack, so he'd have to round the number up.

Solving Problems by Multiplying Fractions by Integers

Think about the problem "How many toes do 5 people have altogether, assuming they each have 10 toes?"

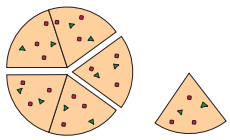
The question is asking how many there are in 5 sets of 10.
To solve it, you **multiply** 5 by 10 to get 50.

You can do problems of this type that contain fractions in the same way.

Example 1

Three people each ate $\frac{2}{5}$ of a pizza at a buffet.

How much pizza did they eat altogether?



Solution

Multiply 3 by the fraction $\frac{2}{5}$.

$$3 \times \frac{2}{5} = \frac{3 \times 2}{5}$$ **Multiply the integer by the numerator**

$$= \frac{6}{5} = 1\frac{1}{5}$$ **Convert to a mixed number**

They ate $1\frac{1}{5}$ **pizzas** altogether.

You might come across a similar problem involving mixed numbers.

Example 2

Pedro is making paper chains for a party. He needs $1\frac{3}{8}$ packs of paper strips for each chain and needs to make 7 chains. How many packs of paper strips will he use?



Solution

Multiply 7 by $1\frac{3}{8}$, since the question is asking you to find 7 sets of $1\frac{3}{8}$.

$$7 \times 1\frac{3}{8} = 7 \times \frac{11}{8}$$ **Convert the mixed number to an improper fraction**

$$= \frac{7 \times 11}{8}$$ **Multiply the integer by the numerator**

$$= \frac{77}{8} = 9\frac{5}{8}$$ He uses $9\frac{5}{8}$ **packs** of paper strips.

Guided Practice

1. Some motorbikes are going to be transported in a trailer.
Each motorbike weighs $\frac{1}{3}$ of a ton. What do four motorbikes weigh?

2. My dog is $\frac{3}{10}$ of a meter tall.
I'm five times taller than my dog. How tall am I?

3. In a relay race, four athletes each run around a track once.
The track is $87\frac{1}{4}$ meters long. What is the total distance run?

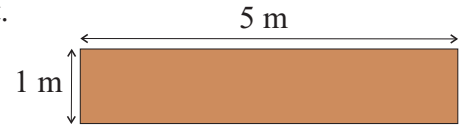
There's another type of problem that can be solved by **multiplying** fractions and integers together. It's less obvious that this type can be solved by multiplication, so it's **really important** that you learn to spot them.

Find a Fraction of an Integer by Multiplying

If you're asked to find a fraction **of** an integer, you **multiply** the fraction and the integer together.

Example 3

A rectangular garden border is 5 m long by 1 m wide. Three-quarters of the border has grass planted on it. What is the area of the grass?



Solution

Find the area of the rectangular border using the **formula**:
area = length × width.

So the area of the border is $5 \times 1 = \mathbf{5\ m^2}$

"$\frac{3}{4}$ **of**" this area is $\frac{3}{4} \times 5\ m^2$.

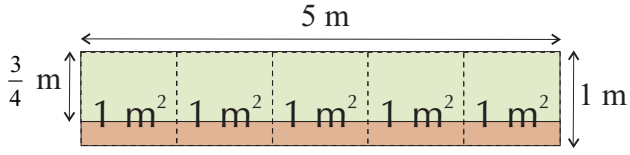
$$\frac{3}{4} \times 5 = \frac{3 \times 5}{4} m^2$$

$$= \frac{15}{4} m^2 = \mathbf{3\frac{3}{4}\ m^2}$$

So, the area of the grass = $\mathbf{3\frac{3}{4}\ m^2}$

Check it out:
Area is covered in more detail in Section 2.3.
For now, you just need to understand what it means when something is multiplied by a fraction.

It's easier to visualize why multiplying works for a problem like this by using a **diagram**. Imagine that the area planted with grass was along one side.



The grass area is $\frac{3}{4}$ **of 5 m²**. You could work it out using the area formula:

area = length × width. So $\frac{3}{4}$ **of 5 m²** = $\frac{3}{4}$ **× 5 m²**.

Check it out:
When you have to find a fraction **of** something, you can replace "of" with a multiplication. So $\frac{3}{4}$ of 5 means $\frac{3}{4} \times 5$, and so on.

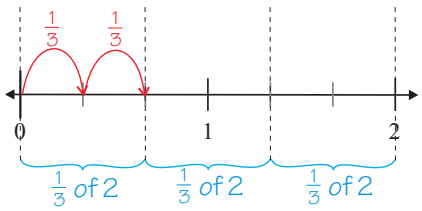
Another Way to Look at This...

From the previous page, you know that $\frac{1}{3} \times 2$ is the same as $\frac{1}{3}$ of 2.

You can also show this on a **number line**:

$\frac{1}{3} \times 2$

$\frac{1}{3}$ of 2



The examples below ask you to find **a fraction of a number**.

Example 4

Shatika, Jose, Kim, and Dan have 5 meters of beading wire. They want to cut it up so that they all have equal lengths. How many meters do they each get? Give your answer as a mixed number.

Solution

They each get $\frac{1}{4}$ of 5 meters. So multiply $\frac{1}{4}$ by 5.

$\frac{1}{4} \times 5 = \frac{1 \times 5}{4}$ **Multiply the numerator by the integer**

$= \frac{5}{4}$ **Convert to a mixed number**

$= 1\frac{1}{4}$ **meters**

Don't forget:
You need to include the correct units in your answer. In this example, it's meters.

Example 5

When comparing test scores, Michael noticed that his score was $\frac{7}{8}$ of Janelle's score. If Janelle's score was 88, what was Michael's score?

Solution

You need to find out $\frac{7}{8}$ of 88. So multiply $\frac{7}{8}$ by 88.

$\frac{7}{8} \times 88 = \frac{7 \times 88}{8}$ **Multiply the numerator by the integer**

$= \frac{616}{8}$ **Simplify the answer**

$= 77$

Check it out:

To find $\frac{7}{8}$ of a number you could instead find $\frac{1}{8}$ of it and multiply it by 7.

$\frac{1}{8}$ of 88 is $88 \div 8 = 11$.

$\frac{7}{8}$ of 88 is $7 \times 11 = 77$.

Guided Practice

4. One palm tree is 25 feet tall. Another tree is $\frac{3}{4}$ of its height. Find the height of the shorter tree.

5. Amber is baking bread. The recipe calls for 800 g of flour, but Amber wants to make her loaf $\frac{5}{8}$ the size of the one in the recipe. How much flour should she use?

Independent Practice

Evaluate Exercises 1–3.

1. $4 \times 5\frac{1}{7}$ **2.** $2\frac{1}{13} \times -6$ **3.** $10 \times 1\frac{1}{10}$

4. Aisha is painting a wall. After the first hour, she has painted $\frac{2}{15}$ of the wall. How much of the wall will she have painted after 4 hours, assuming she continues painting at the same hourly rate?

5. A summer camp knows that each camper will eat sandwiches made from an average of $\frac{3}{16}$ of a loaf of bread. How many loaves of bread should be used to make sandwiches for 121 campers?

6. Adult guinea pigs consume approximately $2\frac{1}{10}$ ounces of dried food and $6\frac{2}{3}$ tablespoons of water each day. How much dried food and water would you expect three adult guinea pigs to consume each day?

Don't forget:
You need to include units for the answers to Exercise 6–9.

7. Juan and Kate are each knitting a scarf. Juan's scarf is 24 inches long so far. If Kate's scarf is $\frac{3}{5}$ of the length of Juan's, how long is it?

8. Connor has 25 liters of water for a six-day camping trip. He wants to drink $\frac{1}{6}$ of the water each day. How much is this?

Now try these:
Lesson 3.2.2 additional questions — p440

9. Chang weighed 7 pounds when he was born. His younger brother weighed $\frac{7}{9}$ of this. How much did Chang's brother weigh?

10. Paul's top pinball score is 87,000. Juanita tells him she scored $\frac{3}{8}$ of his score. What was Juanita's score?

Round Up

In earlier grades you spotted repeated-addition problems and solved them by multiplication. Now you have to also watch out for "a fraction of a number" type problems and solve them by multiplication too. Next Lesson, you'll multiply fractions by other fractions and see how this applies to real life.

Lesson 3.2.3

Multiplying Fractions by Fractions

This Lesson is about multiplying one fraction by another. Not only do you need to know how to do this, but you also have to be able to understand when a problem can be solved by this method.

Multiply Both Tops and Bottoms of Fractions

To multiply one fraction by another, you just multiply the numerators together, then multiply the denominators together. This gives you the numerator and denominator of the product.

For example: $\frac{2}{3}\times\frac{5}{7}=\frac{2\times5}{3\times7}=\frac{10}{21}$

When multiplying two fractions:

multiply both numerators together and both denominators together.

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$$

Example 1

Evaluate i) $\frac{7}{8}\times\frac{5}{6}$ ii) $\frac{2}{3}\times\frac{4}{7}$ iii) $\frac{5}{7}\times\frac{9}{11}$

Solution

i) $\frac{7}{8}\times\frac{5}{6}=\frac{7\times5}{8\times6}=\frac{35}{48}$

ii) $\frac{2}{3}\times\frac{4}{7}=\frac{2\times4}{3\times7}=\frac{8}{21}$

iii) $\frac{5}{7}\times\frac{9}{11}=\frac{5\times9}{7\times11}=\frac{45}{77}$

Multiply both of the numerators together, then multiply both of the denominators together.

If you're asked to multiply a mixed number, just convert it to an improper fraction first.

Example 2

Evaluate $\frac{3}{5}\times2\frac{1}{4}$.

Solution

$\frac{3}{5}\times2\frac{1}{4}=\frac{3}{5}\times\frac{(2\times4)+1}{4}$

Convert the mixed number to an improper fraction.

$=\frac{3}{5}\times\frac{9}{4}=\frac{3\times9}{5\times4}$

Multiply the numerators together and the denominators together.

$=\frac{27}{20}$, or $1\frac{7}{20}$

California Standards:

Number Sense 2.1

Solve problems involving addition, subtraction, **multiplication**, and division **of positive fractions** and explain why a particular operation was used for a given situation.

Number Sense 2.2

Explain the meaning of multiplication and division **of positive fractions and perform the calculations** (e.g., $\frac{5}{8}\div\frac{15}{16}=\frac{5}{8}\times\frac{16}{15}=\frac{2}{3}$).

What it means for you:

You'll learn the skills needed to multiply fractions by other fractions. You'll also solve some real-life problems that involve doing this.

Key words:

- integer
- fraction
- denominator
- numerator
- improper fraction
- mixed number

Don't forget:

A product is the result of multiplying numbers together.

Check it out:

Any integer divided by 1 is equivalent to the integer itself.

For example, $3=\frac{3}{1}$, and $5=\frac{5}{1}$.

So multiplying a fraction by an integer can also be thought of as a fraction-by-fraction multiplication.

So, $3\times\frac{4}{5}=\frac{3}{1}\times\frac{4}{5}=\frac{3\times4}{1\times5}=\frac{12}{5}$.

This is what you were doing in Lessons 3.2.1 and 3.2.2.

Guided Practice

Evaluate the following.

1. $\frac{3}{4} \times \frac{1}{7}$
2. $\frac{4}{9} \times \frac{7}{13}$
3. $\frac{11}{10} \times \frac{7}{9}$
4. $2\frac{2}{3} \times \frac{5}{7}$
5. $\frac{7}{2} \times \frac{7}{5}$
6. $1\frac{2}{3} \times 1\frac{1}{13}$

Multiply to Find a Fraction of a Fraction

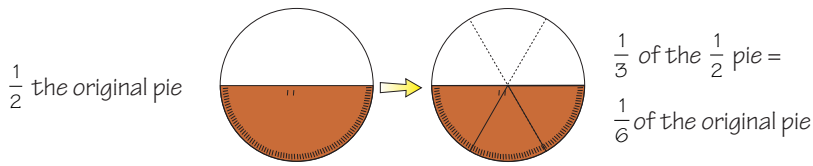
In the previous Lesson, you found a fraction of an integer by multiplying the fraction and integer together. You can use multiplication in the same way to find a fraction of a fraction.

Example 3

You have half a pie left and want to split it into thirds to share with your two friends. What fraction of the original whole pie do you each get?

Solution

This diagram shows you what's going on in this problem:



But you don't need to draw a diagram.

You can find what fraction each person gets, by multiplying $\frac{1}{3}$ by $\frac{1}{2}$.

$$\frac{1}{3} \times \frac{1}{2} = \frac{1 \times 1}{3 \times 2}$$

Multiply the numerators together and the denominators together

$$= \frac{1}{6}$$

So you each get $\frac{1}{6}$ of the original pie.

Check it out:

This works the same with sets of objects as with shapes. For example, if $\frac{1}{2}$ of your class went on a canoe trip, and $\frac{1}{3}$ of these students got wet, then $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ of your class got wet on the trip.

To find a fraction of a mixed number, just change the mixed number to an improper fraction first.

Example 4

You have two and a half grapefruits and are making four fruit salads. You want to put one-quarter of the total amount of grapefuit you have into each fruit salad. How much grapefruit should you put in each?

Solution

You want to find $\frac{1}{4}$ of $2\frac{1}{2}$. So multiply these numbers together.

$$\frac{1}{4} \times 2\frac{1}{2} = \frac{1}{4} \times \frac{5}{2}$$ **Convert the mixed number to an improper fraction**

$$= \frac{1 \times 5}{4 \times 2}$$ **Multiply the numerators together and the denominators together**

$$= \frac{5}{8}$$

So you should put $\frac{5}{8}$ of a grapefruit in each fruit salad.

Guided Practice

7. It takes half an hour for Elise to practice her musical instrument. She spends one-quarter of that time warming up.
What fraction of an hour does she spend warming up?

8. Marsha's creative writing story was one and a half pages long. David's story was two-thirds of the length of Marsha's story.
How long was David's story?

Independent Practice

Evaluate Exercises 1–6.

1. $\frac{3}{8} \times \frac{3}{4}$ **2.** $\frac{2}{5} \times \frac{4}{31}$ **3.** $\frac{2}{9} \times \frac{2}{3}$

4. $2\frac{1}{5} \times \frac{1}{4}$ **5.** $\frac{4}{7} \times 1\frac{4}{7}$ **6.** $\frac{8}{15} \times 1\frac{7}{8}$

7. Lola drank $\frac{3}{4}$ of a cup of tea. The cup of tea contained $1\frac{1}{4}$ teaspoons of sugar. How much sugar did Lola consume?

8. Arthur finished his assignment in $4\frac{1}{3}$ hours.
His classmate finished in half the time that Arthur did.
How much time did it take his classmate to finish?

9. A sponsored walk is $9\frac{3}{4}$ miles. There are refreshments provided two-thirds of the way along the route. After how many miles is this?

Now try these:
Lesson 3.2.3 additional questions — p440

Round Up

Multiplying fractions by fractions is a skill that you'll use time and time again in solving real-life problems and in simplifying algebraic expressions — so it's worth making sure you know how to do it.

Lesson 3.3.1

Section 3.3
Dividing by Fractions

The last few Lessons were all about multiplying fractions. Dividing by fractions is really closely related to this — to be able to divide by fractions, you actually use the skills you learned for multiplication. Also, before you start dividing by fractions, you need to know about reciprocals, so that's where this Lesson starts.

Reciprocals Multiply Together to Give 1

Two numbers that multiply together to give **1** are known as **reciprocals** of each other. Another name for a reciprocal is a **multiplicative inverse**.

Here are some examples of multiplicative inverses:

$$\frac{2}{3} \text{ and } \frac{3}{2} \Rightarrow \frac{2}{3}\times\frac{3}{2} = \frac{2\times3}{3\times2} = \frac{6}{6} = 1$$

$$\frac{4}{5} \text{ and } \frac{5}{4} \Rightarrow \frac{4}{5}\times\frac{5}{4} = \frac{4\times5}{5\times4} = \frac{20}{20} = 1$$

You might have noticed that you "**flip**" or "**invert**" a fraction to get its reciprocal. The **numerator** becomes the **denominator** and the **denominator** becomes the **numerator**.

$$\frac{4}{5} \rightarrow \frac{5}{4} \qquad \frac{2}{3} \rightarrow \frac{3}{2}$$

So, in mathematical terms:

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, for $a \neq 0$ and $b \neq 0$.

Write Integers as Fractions — Then Find the Reciprocals

An **integer** can be written as a fraction with a denominator of 1.

For example, 7 can be written as $\frac{7}{1}$. So its reciprocal is $\frac{1}{7}$.

$$\Rightarrow \frac{7}{1} \rightarrow \frac{1}{7}$$

Similarly, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of 5 is $\frac{1}{5}$.

Guided Practice

Write down the reciprocals of the following.
Show that they are reciprocals by multiplying.

1. $\frac{7}{8}$ **2.** $\frac{9}{7}$ **3.** 8 **4.** 14 **5.** $1\frac{1}{3}$ **6.** $3\frac{3}{4}$

California Standards:

Number Sense 2.1
Solve problems involving addition, subtraction, multiplication, and **division of positive fractions and explain why a particular operation was used for a given situation.**

Number Sense 2.2
Explain the meaning of multiplication and **division of positive fractions and perform the calculations** (e.g., $\frac{5}{8} \div \frac{15}{16} = \frac{5}{8} \times \frac{16}{15} = \frac{2}{3}$).

What it means for you:

You'll learn what it means when you divide by a fraction, and why you can use multiplication to do this. You'll also learn what reciprocals are, because you need to use them to divide by fractions.

Key words:

- denominator
- divide
- multiply
- numerator
- reciprocal

Don't forget:

Check your reciprocals by multiplying. You should always get a product of 1:

$$\frac{7}{1}\times\frac{1}{7} = \frac{7\times1}{1\times7} = \frac{7}{7} = 1$$

Check it out:

To find the reciprocal of a mixed number, first convert it to an improper fraction. For example:

$$2\frac{3}{11} = \frac{(2\times11)+3}{11} = \frac{25}{11}$$

So the reciprocal of $2\frac{3}{11}$ is $\frac{11}{25}$.

To Divide by Any Fraction, Multiply by the Reciprocal

The **general rule** for dividing by a fraction is to **multiply** by the **reciprocal** of the fraction.

So, in math language: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

Example 1

What is $2 \div \frac{1}{4}$?

Solution

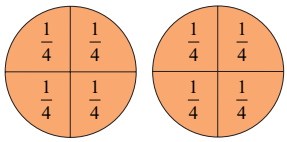
To divide 2 by $\frac{1}{4}$, multiply 2 by the reciprocal of $\frac{1}{4}$.

$2 \div \frac{1}{4} = 2 \times 4$ ← The reciprocal of $\frac{1}{4}$ is 4.

$= \mathbf{8}$

The division $2 \div \frac{1}{4}$ is asking **how many quarters** there are in 2.

Each circle represents one whole unit.



There are 4 quarters in each whole unit, so in 2 whole units there are **$2 \times 4 = 8$ quarters**.

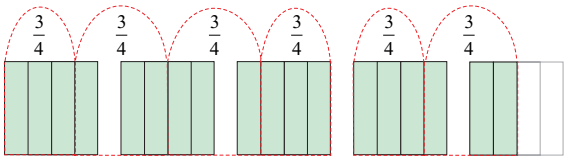
Check it out:

Interpret the division in the same way that you might if it contained integers.
For instance: $9 \div 3$ can be interpreted as asking how many 3s there are in 9.

Example 2

What is $4\frac{1}{2} \div \frac{3}{4}$? Draw a diagram to show what this problem means.

Solution



$4\frac{1}{2} \div \frac{3}{4}$ means how many $\frac{3}{4}$s there are in $4\frac{1}{2}$.

The diagram shows that there are **6**.

You can also get this answer by multiplying $4\frac{1}{2}$ by the reciprocal of $\frac{3}{4}$:

$4\frac{1}{2} \div \frac{3}{4} = 4\frac{1}{2} \times \frac{4}{3} = \frac{9}{2} \times \frac{4}{3}$

Reciprocal of $\frac{3}{4}$

Change the division to multiplication by the reciprocal.

$= \frac{9 \times 4}{2 \times 3}$

Multiply the fractions.

$= \frac{36}{6} = \mathbf{6}$

Don't forget:

When you multiply a mixed number, change it to an improper fraction first.

So, $4\frac{1}{2} = \frac{(4 \times 2) + 1}{2}$

Guided Practice

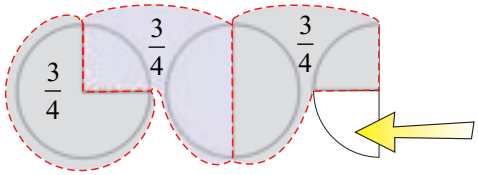
Draw a diagram to represent each of the divisions in Exercises 7–10. Then find the answers.

7. $5 \div \frac{5}{9}$ **8.** $3\frac{1}{2} \div \frac{1}{4}$ **9.** $2\frac{1}{4} \div \frac{3}{8}$ **10.** $2\frac{1}{2} \div \frac{5}{6}$

The Solution May Be a Fraction

When you divide by a fraction, you might get a bit left over.

For example, think about the division $2\frac{1}{2} \div \frac{3}{4}$:



$\frac{1}{4}$ left over — that's $\frac{1}{3}$ of $\frac{3}{4}$

The diagram shows that $2\frac{1}{2}$ contains **three complete** $\frac{3}{4}$s, and also $\frac{1}{3}$ of $\frac{3}{4}$.

So $2\frac{1}{2} \div \frac{3}{4} = 3\frac{1}{3}$.

You could, of course, have done this division by multiplying by the reciprocal. The examples below are done in this way.

Check it out:

$$2\tfrac{1}{2} \div \tfrac{3}{4} = \tfrac{5}{2} \div \tfrac{3}{4} = \tfrac{5}{2} \times \tfrac{4}{3} = \tfrac{20}{6} = \tfrac{10}{3} = 3\tfrac{1}{3}$$

Example 3

What is $\frac{1}{2} \div \frac{2}{3}$?

Solution

$$\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \times \frac{3}{2}$$

Multiply by the reciprocal of the fraction you are dividing by.

$$= \frac{1 \times 3}{2 \times 2} = \frac{3}{4}$$

Check it out:

You're dividing $\frac{1}{2}$ by something bigger than $\frac{1}{2}$ — so the answer is less than 1.

Divide a fraction by an **integer** in exactly the same way — multiply by its reciprocal.

Example 4

What is $1\frac{1}{4} \div 4$?

Solution

$$1\tfrac{1}{4} \div 4 = \frac{5}{4} \div 4$$

Convert the mixed number to an improper fraction.

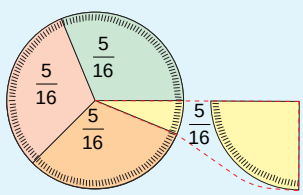
$$= \frac{5}{4} \times \frac{1}{4}$$

Multiply by the reciprocal of the integer you are dividing by.

$$= \frac{5 \times 1}{4 \times 4} = \frac{5}{16}$$

Check it out:

$1\frac{1}{4} \div 4$ could be pictured as sharing out $1\frac{1}{4}$ pies between four people. Each person gets $\frac{5}{16}$ of a pie.



Guided Practice

Draw diagrams to show the divisions in Exercises 11–13.

11. $\frac{1}{2} \div 3$ **12.** $\frac{1}{4} \div 2$ **13.** $1\frac{1}{3} \div 3$

Use multiplication to find the solutions to Exercises 14–19.

14. $\frac{1}{3} \div \frac{1}{2}$ **15.** $\frac{1}{15} \div \frac{2}{7}$ **16.** $\frac{7}{8} \div \frac{3}{7}$

17. $\frac{2}{7} \div 2\frac{1}{2}$ **18.** $3\frac{2}{3} \div 3\frac{7}{8}$ **19.** $\frac{1}{15} \div 2$

Independent Practice

Give the reciprocals of the numbers given in Exercises 1–8.

1. $\frac{7}{11}$ **2.** $\frac{7}{3}$ **3.** 3 **4.** $\frac{1}{6}$

5. 13 **6.** $3\frac{3}{4}$ **7.** $1\frac{8}{9}$ **8.** $7\frac{1}{6}$

Check it out:
Make sure you don't multiply the reciprocals of both fractions together — it's only the divisor that becomes its reciprocal.

Draw diagrams to represent the problems in Exercises 9–11.

9. $4 \div \frac{2}{5}$ **10.** $6 \div \frac{1}{3}$ **11.** $2\frac{1}{4} \div \frac{1}{8}$

Use multiplication to find the solutions to Exercises 12–17.

12. $8 \div \frac{1}{5}$ **13.** $12 \div \frac{2}{3}$ **14.** $1\frac{2}{3} \div \frac{5}{6}$

15. $\frac{3}{4} \div \frac{3}{8}$ **16.** $5\frac{1}{3} \div 1\frac{1}{2}$ **17.** $\frac{3}{10} \div 4$

18. Is it possible to divide a mixed number by an integer to produce:
a) a number greater than 1?
b) 1?
c) a number less than 1?
Explain your reasoning and give an example of each where possible.

19. Write the expression $y \div z$ as a product.

20. When Nancy divided a fraction by 2, she got $\frac{1}{16}$ for her answer. What fraction was she dividing?

21. When Ken divided $\frac{7}{8}$ by an integer, he got $\frac{7}{32}$. What integer was he dividing by?

Now try these:
Lesson 3.3.1 additional questions — p440

Round Up

After this Lesson, you should be able to understand what divisions involving fractions mean — this'll help you recognize when you should divide by a fraction in real-life problems. You'll meet some of these in the next Lesson.

Lesson 3.3.2

Solving Problems by Dividing Fractions

California Standard:

Number Sense 2.1

Solve problems involving addition, subtraction, multiplication, and **division of positive fractions and explain why a particular operation was used for a given situation.**

What it means for you:

You'll practice solving lots of real-life problems that involve dividing fractions.

Key words:

- divide
- multiply
- reciprocal

Check it out:

Don't be scared by the fractions in these problems. If these problems had contained whole numbers, you'd solve them in exactly the same way — by division. For instance, imagine Example 1 had involved an 8-foot pipe being divided into sections 2 feet long. You'd work out the number of sections you'd get by calculating 8 ÷ 2.

In the last Lesson, you practiced doing divisions involving fractions and mixed numbers. In this Lesson, you'll use these skills to tackle real-life word problems. It's really important that you learn to recognize the types of problems that use division of fractions.

Division Problems Often Involve Partitioning Something

If you come across a problem that involves splitting something up into equal parts, it often means you have to use division to solve it.

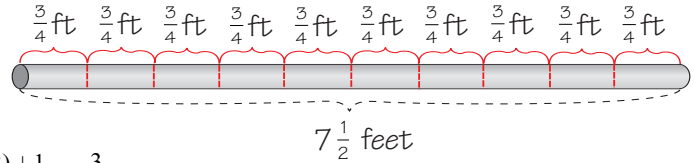
Example 1

A pipe $7\frac{1}{2}$ feet long is divided into sections $\frac{3}{4}$ of a foot long.

How many sections are made from the pipe?

Solution

Divide $7\frac{1}{2}$ by $\frac{3}{4}$:



$$7\frac{1}{2} \div \frac{3}{4} = \frac{(7\times 2)+1}{2} \div \frac{3}{4}$$

Convert the mixed number to an improper fraction.

$$= \frac{15}{2} \div \frac{3}{4}$$

$$= \frac{15}{2} \times \frac{4}{3}$$

Multiply by the reciprocal of the fraction you are dividing by.

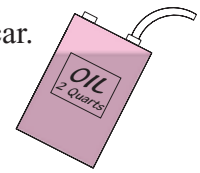
$$= \frac{15\times 4}{2\times 3} = \frac{60}{6} = 10$$

So you get 10 sections of pipe.

Example 2

Ben has $1\frac{7}{8}$ quarts of motor oil. He puts half of it in his car.

How much oil did he put in his car?



Solution

You have to find out what half of $1\frac{7}{8}$ quarts is. So you divide it by 2.

$$1\frac{7}{8} \div 2 = \frac{(1\times 8)+7}{8} \div 2$$

Convert the mixed number to an improper fraction.

$$= \frac{15}{8} \div 2$$

$$= \frac{15}{8} \times \frac{1}{2}$$

Multiply by the reciprocal of the integer you are dividing by.

$$= \frac{15\times 1}{8\times 2} = \frac{15}{16} \text{ quarts}$$

Guided Practice

1. Dolores had $\frac{2}{3}$ of a gallon of apple juice.
She shares it equally between herself and two friends.
What fraction of a gallon does each person get?

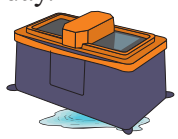
2. TJ is knitting squares that will be sewn together to make blankets for an overseas charity. He has set aside $18\frac{3}{4}$ hours over the next month to knit squares. It takes him $1\frac{1}{4}$ hours to knit one square.
How many squares will he have time to knit?

Division Can Be Used for Repeated Subtraction Too

Another type of division problem is the **repeated subtraction** type.

Example 3

A large water tank contains 600 gallons of water.
The tank has a leak and loses $\frac{2}{3}$ of a gallon of water each day.
After how many days will the tank be empty?



Solution

You need to figure out how many "$\frac{2}{3}$ of a gallon" there are in the tank.
So divide the number of gallons in the tank by $\frac{2}{3}$.

$$600 \div \frac{2}{3} = 600 \times \frac{3}{2}$$

Multiply by the reciprocal of the fraction you are dividing by.

$$= \frac{600 \times 3}{2}$$

$$= 900$$

The tank will be empty after 900 days.

Check it out:
"Repeated subtraction" and "partitioning" problems are very similar. Exactly the same methods are used to solve them.

Guided Practice

3. A bowl contains 4 liters of fruit punch.
How many students can fill $\frac{1}{5}$-liter cups from the bowl?

4. The campers in each cabin at a summer camp want to make a welcome sign. There is a roll of paper with $14\frac{1}{2}$ meters of paper on it.
How many $1\frac{1}{5}$-meter-long signs can be made from the roll?

Check it out:
Think carefully about the answer to Guided Practice Exercise 4. The question asks, "How many cabins?" It wouldn't make sense to have a fraction of a cabin in your answer.

Don't Forget — Division Is the Inverse of Multiplication

Some problems might be tricky to spot as **divisions** immediately. It could be easier to write a **multiplication equation** containing a **variable**, and then solve it.

Example 4

After $7\frac{1}{3}$ miles on a sponsored dog walk, there is a stand giving out dog cookies. This is $\frac{3}{4}$ of the way along the route.

How long is the whole route?

Solution — Method 1

Let d represent the length of the whole route.

$\frac{3}{4}$ of $d = 7\frac{1}{3}$ miles

So, $\frac{3}{4} \times d = 7\frac{1}{3}$

To solve this equation, you divide both sides by $\frac{3}{4}$ to get d by itself.

$d = 7\frac{1}{3} \div \frac{3}{4}$ — **Get the variable d by itself.**

$d = \frac{(7\times3)+1}{3} \div \frac{3}{4} = \frac{22}{3} \div \frac{3}{4}$ — **Convert the mixed number to an improper fraction.**

$d = \frac{22}{3} \times \frac{4}{3}$ — **Multiply by the reciprocal of the fraction you are dividing by.**

$d = \frac{88}{9} = 9\frac{7}{9}$

The route is $9\frac{7}{9}$ miles long.

Don't forget:
To find a fraction of a number, you multiply the fraction by the number.

Another way to do this would be to find out how many miles $\frac{1}{4}$ of the route is, then **multiply** this distance by **four** to find the length of the whole route.

Solution — Method 2

$7\frac{1}{3}$ miles is equal to $\frac{3}{4}$ of the route.

So $\frac{1}{4}$ of the route $= 7\frac{1}{3} \div 3 = \frac{22}{3} \div 3$

$= \frac{22}{3} \times \frac{1}{3} = \frac{22}{9}$ miles

The whole route is equal to four-quarters.

So the whole route $= \frac{22}{9} \times 4 = \frac{22\times4}{9} = \frac{88}{9} = 9\frac{7}{9}$ **miles**

Don't forget:
This method involves multiplying a fraction by an integer. You do this by multiplying the numerator by the integer and keeping the denominator the same.

Guided Practice

5. After canoeing 19 miles down the river, I was disappointed to find out that I was only $\frac{4}{9}$ of the way to my destination. How far was I expecting to canoe in total?

6. A fish tank has $13\frac{1}{3}$ gallons of water emptied into it. This makes it $\frac{3}{8}$ full. How many gallons of water does the tank hold altogether?

Independent Practice

Which of the problems in Exercises 1–2 would you solve by division? Explain your answers.

1. Mrs. Garcia assigned 8 math problems for homework. Each problem takes about $2\frac{1}{2}$ minutes to solve. How long might it take for a student to complete all eight problems?

2. The art teacher has $8\frac{1}{2}$ pounds of clay that he wants to distribute evenly to 12 students. How much clay will each student get?

3. A very tall building has an elevator which is $\frac{1}{4}$ of a mile above the ground when it is at the top floor. What fraction of a mile above the ground is the elevator when it is halfway up the building?

4. Judy fills $2\frac{1}{2}$ buckets with sand. It takes a total of 30 shovel-scoops of sand to do this. What fraction of a bucketful of sand does her shovel hold?

5. Mr. Ruiz has exactly $1\frac{2}{3}$ cups of coffee. He drank the entire amount in 19 sips. What was the average amount of coffee in each sip?

6. A wall is $4\frac{3}{4}$ feet long. The wall is to be divided up into 3 equal sections that will each be painted a different color. How wide will each section be?

7. DeAndre was running in a race. After he had run $4\frac{1}{2}$ miles, he was $\frac{3}{8}$ of the way to the finish. How long was the race?

Now try these:
Lesson 3.3.2 additional questions — p441

Round Up

So that's how division of fractions can be used to solve some types of real-life problems. You've now finished looking at multiplication and division of fractions. In the next few Lessons, you'll learn the techniques for adding and subtracting fractions and how to give answers in their simplest forms.

Section 3.4 introduction — an exploration into:

Adding Fractions: an Area Model

Graph paper is very useful when you are doing fraction addition. The squares on the graph paper let you make realistic "models" of fractions, which you can then add by counting squares. It's a good way to see what it actually means to add fractions.

To **add** two fractions, start by drawing **three identical rectangles**:
— the **heights** of the rectangles should be equal to one of the **denominators**,
— the **widths** should be equal to the **other denominator**.

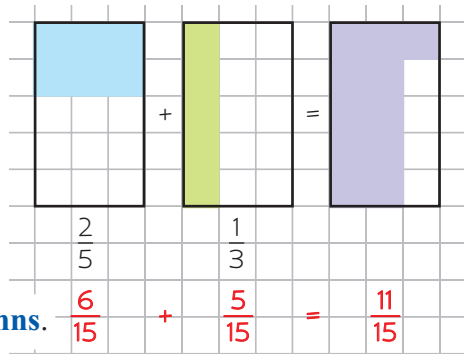
The **denominator** is the number on the **bottom** of a fraction.

Example

Use graph paper to add $\frac{2}{5}+\frac{1}{3}$.

Solution

- Draw three rectangles on graph paper, **5 squares high** and **3 squares wide**.
- Show $\frac{2}{5}$ in the first rectangle by shading **2 of the 5 rows**.
- Show $\frac{1}{3}$ in the next rectangle by shading **1 of the 3 columns**.
- Count the **total** number of shaded boxes in the first two rectangles. Shade this number of boxes in the **third** rectangle.
- **Rewrite** each of the three fractions using the following formula: $\frac{\text{number of shaded boxes}}{\text{total number of boxes}}$



You can use exactly the same method to add fractions that sum to more than 1.

Example

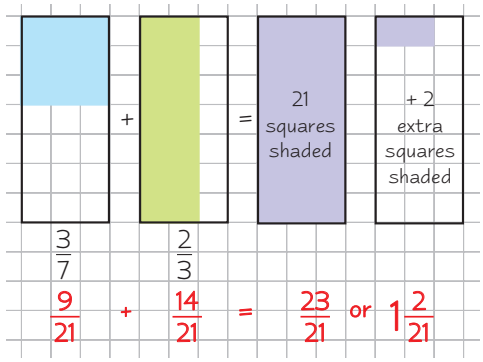
Use graph paper to add $\frac{3}{7}+\frac{2}{3}$.

Solution

Do everything exactly as before, but draw an **extra** rectangle if your answer does not fit.

You can write your answer as:

- an **improper fraction** — such as $\frac{23}{21}$,
- a **mixed number** — such as $1\frac{2}{21}$.



Exercises

1. Do these additions by drawing rectangles on graph paper to represent the fractions.

a. $\frac{1}{4}+\frac{2}{3}$ **b.** $\frac{1}{2}+\frac{2}{5}$ **c.** $\frac{3}{4}+\frac{2}{5}$ **d.** $\frac{5}{6}+\frac{1}{3}$

Round Up

Drawing pictures of fractions on graph paper is a good way to add them.
Think about these pictures while you're learning about the ideas and methods in this Section.

Lesson 3.4.1

Section 3.4
Making Equivalent Fractions

You've met equivalent fractions before — they're the ones that look different but sit at the same place on the number line. This Lesson is all about how you can make new equivalent fractions.

California Standard:

Number Sense 2.1
Solve problems involving addition, subtraction, multiplication, and division of **positive fractions** and explain why a particular operation was used for a given situation.

What it means for you:
You'll learn how to make new equivalent fractions — including ones that are simpler.

Key words:
- equivalent fraction
- simpler
- factor
- cancel
- prime factorization

Don't forget:

You can check the value of a fraction by doing a division — either on a calculator or by using long division.

$\frac{6}{8} = 6 \div 8 = 0.75$

$\frac{9}{12} = 9 \div 12 = 0.75$

$\frac{24}{32} = 24 \div 32 = 0.75$

Make Equivalent Fractions by Multiplying by 1

Equivalent fractions **look different** but have the **same value**, like $\frac{1}{2}$ and $\frac{3}{6}$.

You form equivalent fractions by multiplying by a **fraction equal to 1** — that is, by a fraction whose **numerator and denominator are equal**.

Example 1

By multiplying by a suitable fraction, show that $\frac{3}{6}$ is equivalent to $\frac{1}{2}$.

Solution

Any fraction with the **same numerator and denominator** is **equal to 1**. This means you can multiply $\frac{1}{2}$ by $\frac{3}{3}$ and get a fraction with equal value.

$$\frac{1}{2} \times \frac{3}{3} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$

So the two fractions are **equivalent**.

Example 2

Find three fractions equivalent to $\frac{3}{4}$.

Solution

Pick any three fractions equivalent to 1, for example $\frac{2}{2}$, $\frac{3}{3}$, and $\frac{8}{8}$. Then multiply $\frac{3}{4}$ by each of them in turn.

1. $\frac{3}{4} \times \frac{2}{2} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$

2. $\frac{3}{4} \times \frac{3}{3} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$

3. $\frac{3}{4} \times \frac{8}{8} = \frac{3 \times 8}{4 \times 8} = \frac{24}{32}$

You sometimes have to choose your fractions very carefully.

Example 3

Fill in the missing numbers in these fractions: $\frac{1}{2} = \frac{a}{8} = \frac{32}{b}$

Solution

To get 8 as the denominator in $\frac{a}{8}, \frac{1}{2}$ must have been multiplied by $\frac{4}{4}$.

$\frac{1}{2} \times \frac{4}{4} = \frac{1 \times 4}{2 \times 4} = \frac{4}{8}$, so $a = 4$.

To get 32 as the numerator in $\frac{32}{b}$, $\frac{1}{2}$ must have been multiplied by $\frac{32}{32}$.

$\frac{1}{2} \times \frac{32}{32} = \frac{1 \times 32}{2 \times 32} = \frac{32}{64}$, so $b = 64$.

So $\frac{1}{2} = \frac{4}{8} = \frac{32}{64}$

Guided Practice

1. Find three fractions equivalent to $\frac{3}{5}$.

2. By multiplying by a suitable fraction, show that $\frac{1}{5}$ and $\frac{7}{35}$ are equal.

Fill in the missing numbers in Exercises 3–4.

3. $\frac{2}{3} = \frac{4}{6} = \frac{?}{9} = \frac{?}{12} = \frac{10}{?} = \frac{12}{?} = \frac{?}{900}$

4. $\frac{4}{5} = \frac{8}{10} = \frac{?}{15} = \frac{48}{?} = \frac{?}{300} = \frac{480}{?}$

Make Fractions Simpler by Canceling

Canceling is a way of making equivalent fractions that are **simpler**.

Instead of **multiplying** the top and bottom lines by the same number, canceling involves **dividing** them.

Example 4

Form a simpler fraction equal to $\frac{10}{15}$.

Solution

To cancel a fraction, you need to look for numbers that are **factors** of both the **numerator** and **denominator**.

Here, **5** divides into the numerator and denominator: $\frac{10}{15} = \frac{2 \times 5}{3 \times 5}$

You can **cancel** common factors — so $\frac{2 \times \cancel{5}}{3 \times \cancel{5}} = \frac{2}{3}$

Check it out:
When you cancel a fraction, the numbers get smaller in size and easier to use.

Don't forget:
A factor of a number divides evenly into that number. For example, 5 is a factor of 10.

Guided Practice

5. By canceling, find a simpler fraction equivalent to $\frac{6}{8}$.

Cancel the fractions in Exercises 6–11 to find simpler forms.

6. $\frac{14}{16}$ **7.** $\frac{15}{35}$ **8.** $\frac{44}{55}$

9. $\frac{14}{21}$ **10.** $\frac{9}{15}$ **11.** $\frac{26}{39}$

Don't forget:
A prime factorization is when you write a number as a product of prime factors, for example:
$10 = 5 \times 2$
$8 = 2 \times 2 \times 2 = 2^3$

Check it out:
In fact, the fraction $\frac{12}{15}$ is a simpler form of $\frac{24}{30}$, and so is a possible answer in Example 5. But it's not the simplest form — and usually, the simpler the fraction, the better. There's more about the simplest form in Lesson 3.4.2.

You Can Cancel More Than One Common Factor

You can cancel as many common factors as you can find.

Example 5

Simplify the fraction $\frac{24}{30}$.

Solution

There are different ways to do this.

- You can look for the **biggest common factor**, then cancel it. $\frac{24}{30} = \frac{4 \times \cancel{6}}{5 \times \cancel{6}} = \frac{4}{5}$
- You can write down the **prime factorizations** of the numerator and denominator and cancel **all** the common factors. $\frac{24}{30} = \frac{2 \times 2 \times \cancel{2} \times \cancel{3}}{5 \times \cancel{2} \times \cancel{3}} = \frac{4}{5}$
- You can do it **in stages**. You can cancel one common factor first. $\frac{24}{30} = \frac{12 \times \cancel{2}}{15 \times \cancel{2}} = \frac{12}{15}$

 Then you can cancel another common factor. $\frac{12}{15} = \frac{4 \times \cancel{3}}{5 \times \cancel{3}} = \frac{4}{5}$

All three methods give the same answer.

Guided Practice

12. Find two simpler fractions that are equivalent to $\frac{8}{48}$.

13. By canceling, find a simpler fraction equivalent to $\frac{21}{56}$.

Fill in the missing numbers in Exercises 14–18.

14. $\frac{80}{60} = \frac{40}{30} = \frac{?}{15} = \frac{4}{?}$

15. $\frac{150}{270} = \frac{?}{135} = \frac{15}{?} = \frac{5}{?}$

16. $\frac{108}{126} = \frac{36}{?} = \frac{12}{?} = \frac{?}{7}$

17. $\frac{420}{490} = \frac{60}{?} = \frac{?}{14} = \frac{?}{7}$

18. $\frac{180}{225} = \frac{60}{?} = \frac{?}{25} = \frac{4}{?}$

Independent Practice

Now try these:
Lesson 3.4.1 additional questions — p441

In Exercises 1–8, find two fractions equivalent to the one given.

1. $\frac{1}{9}$ 2. $\frac{3}{4}$ 3. $\frac{6}{7}$ 4. $\frac{1}{3}$

5. $\frac{2}{3}$ 6. $\frac{4}{7}$ 7. $\frac{1}{11}$ 8. $\frac{5}{8}$

Fill in the missing numbers in Exercises 9–15.

9. $\frac{7}{12} = \frac{14}{?} = \frac{21}{?} = \frac{?}{48} = \frac{?}{60}$

10. $\frac{?}{8} = \frac{2}{4} = \frac{?}{2} = \frac{36}{?} = \frac{?}{80}$

11. $\frac{18}{72} = \frac{6}{?} = \frac{9}{?} = \frac{?}{4}$

12. $\frac{6}{9} = \frac{?}{18} = \frac{18}{?} = \frac{2}{?} = \frac{54}{?}$

13. $\frac{?}{120} = \frac{10}{?} = \frac{50}{?} = \frac{2}{3}$

14. $\frac{40}{?} = \frac{?}{18} = \frac{8}{9} = \frac{?}{63} = \frac{?}{27} = \frac{32}{?} = \frac{128}{?} = \frac{96}{?}$

15. $\frac{?}{63} = \frac{?}{42} = \frac{20}{?} = \frac{95}{?} = \frac{10}{?} = \frac{?}{35} = \frac{15}{?} = \frac{5}{7}$

Round Up

There are two tricks to finding equivalent fractions. The first is easy — pick any integer and multiply the top and bottom lines of the fraction by it. The second is harder — you have to find something that both parts of the fraction can be divided by. But if you can, then you'll be rewarded with a simpler fraction.

Lesson 3.4.2

Finding the Simplest Form

California Standard:

Number Sense 2.4
Determine the least common multiple and the **greatest common divisor of whole numbers; use them to solve problems with fractions** (**e.g.,** to find a common denominator to add two fractions or **to find the reduced form for a fraction**).

What it means for you:
You'll learn about the simplest form of a fraction and see a couple of ways of finding it.

Key words:
- simplify
- greatest common divisor
- prime factorization

Check it out:
Some people say "reduced form" instead of "simplest form."

Last Lesson you saw how to make equivalent fractions and how some fractions can be made simpler. This Lesson is all about taking "simpler fractions" as far as you possibly can — to find the simplest form.

Fractions Have a Simplest Form

A fraction like $\frac{1}{2}$ can't be made any simpler because there isn't anything that both 1 and 2 can be divided by.

When a fraction can't be made any simpler, it is in its **simplest form**.

One way to find the simplest form is to keep making a fraction simpler until you can't go any further.

Example 1

Find the simplest form of $\frac{66}{99}$.

Solution

Both 66 and 99 have a factor of 3, so $\frac{66}{99} = \frac{22 \times \cancel{3}}{33 \times \cancel{3}} = \frac{22}{33}$

But 22 and 33 both divide by 11, so $\frac{22}{33} = \frac{2 \times \cancel{11}}{3 \times \cancel{11}} = \frac{2}{3}$

Since nothing divides both 2 and 3, the simplest form of $\frac{66}{99}$ is $\frac{2}{3}$.

Example 2

Find the simplest form of $\frac{84}{126}$.

Solution

Just keep looking for **factors** of **both** the numerator and denominator.

$$\frac{84}{126} = \frac{42 \times \cancel{2}}{63 \times \cancel{2}} = \frac{42}{63} = \frac{14 \times \cancel{3}}{21 \times \cancel{3}} = \frac{14}{21} = \frac{2 \times \cancel{7}}{3 \times \cancel{7}} = \frac{2}{3}$$

Guided Practice

Find the simplest form of the fractions given in Exercises 1–8.

1. $\frac{4}{6}$ **2.** $\frac{20}{35}$ **3.** $\frac{9}{81}$ **4.** $\frac{12}{30}$

5. $\frac{75}{90}$ **6.** $\frac{32}{176}$ **7.** $\frac{135}{195}$ **8.** $\frac{19}{133}$

Greatest Common Divisor Is a Faster Way to Simplify

The **greatest common divisor** of two numbers is the **greatest number** that they can both be divided by.

Don't forget:
A prime factorization is when you "break a number down" into its prime factors.
For example, 15 can be broken down like this:
15 = 5 × 3.
But it can't be broken down any further because both 5 and 3 are prime — nothing divides into 5 except 5 and 1, and nothing divides into 3 except 3 and 1.

To find the greatest common divisor, turn both numbers into **prime factorizations**. Then identify any numbers that are in both factorizations.

$24 = 2 \times 2 \times 2 \times 3$
$84 = 7 \times 2 \times 2 \times 3$
24 and 84 both have factors of 2 (twice) and 3

Multiply the common factors together to get the **greatest common divisor**.

So the greatest common divisor of 24 and 84 is **2 × 2 × 3 = 12**.

Example 3

Find the greatest common divisor of 30 and 42.

Solution

$30 = 2 \times 3 \times 5$
$42 = 2 \times 3 \times 7$
30 and 42 both have factors of 2 and 3

So the greatest common divisor of 30 and 42 is 2 × 3 = 6.

Dividing the numerator and denominator of a fraction by their greatest common divisor is the same as canceling all the common factors at once.

Check it out:
After you divide two numbers by their greatest common divisor, there is nothing else you can divide both by.

This means you can reduce a fraction to its simplest form in a single step.

From above, 12 is the greatest common divisor of 24 and 84.

So $\frac{24}{84} = \frac{2 \times \cancel{12}}{7 \times \cancel{12}} = \frac{2}{7}$, and this must be the simplest form of $\frac{24}{84}$.

Example 4

Find the simplest form of: a) $\frac{30}{42}$, b) $\frac{270}{720}$.

Solution

a) From above, the greatest common divisor of 30 and 42 is 6.

So $\frac{30}{42}$ has simplest form: $\frac{30}{42} = \frac{5 \times 6}{7 \times 6} = \frac{5}{7}$

b) First find the greatest common divisor of 270 and 720.

$$\frac{270}{720} = \frac{2 \times 3 \times 3 \times 3 \times 5}{2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5}$$

(3 and 3 and 5)

270 and 720 both have factors of 2, 3 (twice), and 5.

So the greatest common divisor is 2 × 3 × 3 × 5 = 90.

Therefore $\frac{270}{720}$ has simplest form: $\frac{270}{720} = \frac{3 \times 90}{8 \times 90} = \frac{3}{8}$

Guided Practice

Find the greatest common divisor for each pair of numbers given in Exercises 9–16.

9. 25, 15 **10.** 38, 42 **11.** 60, 42 **12.** 49, 35
13. 27, 99 **14.** 66, 15 **15.** 55, 45 **16.** 47, 37

Find the simplest form of the fractions given in Exercises 17–24.

17. $\frac{2}{12}$ **18.** $\frac{14}{21}$ **19.** $\frac{18}{24}$ **20.** $\frac{42}{70}$

21. $\frac{33}{15}$ **22.** $\frac{24}{30}$ **23.** $\frac{147}{231}$ **24.** $\frac{30}{105}$

Independent Practice

Now try these:
Lesson 3.4.2 additional questions — p441

Find the greatest common divisor for each pair of numbers given in Exercises 1–8.

1. 33, 42 **2.** 38, 57 **3.** 42, 70 **4.** 39, 65
5. 8, 52 **6.** 36, 90 **7.** 18, 27 **8.** 40, 16

9. Simplify the following fractions.
Then arrange them in order of size from least to greatest.

$$\frac{10}{15} \quad \frac{4}{18} \quad \frac{8}{20} \quad \frac{30}{165}$$

10. Which fraction shown below is in its simplest form?

$$\frac{12}{38} \quad \frac{11}{14} \quad \frac{3}{21} \quad \frac{16}{46}$$

Write each of the fractions given in Exercises 11–18 in its simplest form.

11. $\frac{8}{28}$ **12.** $\frac{14}{30}$ **13.** $\frac{16}{32}$ **14.** $\frac{9}{33}$

15. $\frac{68}{110}$ **16.** $\frac{2}{186}$ **17.** $\frac{58}{78}$ **18.** $\frac{120}{180}$

19. Juan scored 22 out of 26 possible points on his history quiz.
What is his score as a reduced fraction?

Round Up

All fractions have a simplest form, and one way to find it is with the greatest common divisor. Finding a fraction's simplest form can seem like a lot of trouble. However, once you've done it, the numbers are smaller. That makes it easier to use in other calculations.

Lesson 3.4.3

Fraction Sums

California Standard:

Number Sense 2.1
Solve problems involving addition, subtraction, multiplication, and division **of positive fractions** and explain why a particular operation was used for a given situation.

What it means for you:

As an introduction to adding and subtracting fractions you'll see how to add and subtract fractions with the same denominator.

Key words:

- fraction
- add
- subtract
- denominator

Check it out:

If you're finding it difficult to understand the logic here, imagine the question was different — for example:
"If you have two pens and you add an extra one pen, then how many pens do you end up with?"
You'd have 3 pens, just like you have 3 fifths in the example.

Adding and subtracting fractions can involve a little more work than multiplying them. This Lesson is about adding and subtracting fractions with the same denominator.

Add or Subtract Numerators If Denominators Are Equal

The denominator of a fraction is similar to the units in a measurement — it tells you **what type** of quantity you are dealing with.
Then the numerator tells you **how many** you have. For example,
$\frac{2}{5}$ means: (i) you are dealing with **fifths**, and (ii) you have **two** of them.

Treating fractions like this can make adding and subtracting them easier.

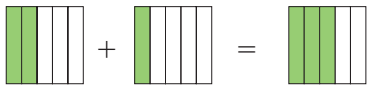
Example 1

Find $\frac{2}{5}+\frac{1}{5}$.

Solution

The sum $\frac{2}{5}+\frac{1}{5}$ is asking: "If you **have two fifths** and you **add an extra one fifth**, then how many fifths do you end up with?"

The answer is $\frac{3}{5}$, as you can see in the diagram.



You can write this as math: $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$

You add or subtract fractions with the **same denominator** by **adding or subtracting the numerators**, while keeping the denominator the same.

For example, $\frac{4}{7}+\frac{1}{7}=\frac{4+1}{7}=\frac{5}{7}$ and $\frac{4}{7}-\frac{1}{7}=\frac{4-1}{7}=\frac{3}{7}$.

Example 2

Find: (i) the sum $\frac{6}{11}+\frac{2}{11}$, and (ii) the difference $\frac{6}{11}-\frac{2}{11}$.

Solution

(i) $\frac{6}{11}+\frac{2}{11}=\frac{6+2}{11}=\frac{8}{11}$ (ii) $\frac{6}{11}-\frac{2}{11}=\frac{6-2}{11}=\frac{4}{11}$

Guided Practice

Work out the sums and differences in Exercises 1–8.

1. $\frac{2}{7}+\frac{1}{7}$ **2.** $\frac{19}{29}-\frac{11}{29}$ **3.** $\frac{50}{6}-\frac{15}{6}$

4. $\frac{6}{17}+\frac{9}{17}$ **5.** $\frac{26}{62}+\frac{15}{62}$ **6.** $\frac{31}{14}-\frac{16}{14}$

7. $\frac{29}{4}-\frac{13}{4}-\frac{1}{4}$ **8.** $\frac{6}{7}-\frac{6}{7}+\frac{6}{7}$

Check it out:
Do additions and subtractions from left to right — just like with integers.
See Lesson 2.1.4 for more information.

You May Need to Simplify Afterwards

Even if you add two fractions that are in their simplest form, you might be able to simplify the answer.

Don't forget:
See Lesson 3.4.2 to remind yourself about how to simplify fractions.

Example 3

Find $\frac{1}{4}+\frac{1}{4}$. Give your answer in its simplest form.

Solution

Add the fractions: $\frac{1}{4}+\frac{1}{4}=\frac{1+1}{4}=\frac{2}{4}$

Simplify the result: $\frac{2}{4}=\frac{1\times 2}{2\times 2}=\frac{1}{2}$

Example 4

What is $\frac{33}{105}+\frac{37}{105}$? Simplify your answer by finding the greatest common divisor of the resulting fraction's numerator and denominator.

Solution

Add the numerators and keep the denominator the same.

$$\frac{33}{105}+\frac{37}{105}=\frac{33+37}{105}=\frac{70}{105}$$

To simplify, find the greatest common divisor of 70 and 105.
Since $70 = 2 \times 5 \times 7$ and $105 = 3 \times 5 \times 7$, the greatest common divisor of 70 and 105 is $5 \times 7 = 35$.

So divide the numerator and denominator by 35 to find the simplest form.

$$\frac{70}{105}=\frac{2\times 35}{3\times 35}=\frac{2}{3}$$

Don't forget:
See Lesson 3.4.2 for more about greatest common divisors.

Guided Practice

Work out the sums and differences in Exercises 9–20, giving your answers in their simplest form.

9. $\frac{7}{8}-\frac{5}{8}$ **10.** $\frac{1}{2}+\frac{1}{2}$ **11.** $\frac{5}{20}+\frac{10}{20}$

12. $\frac{5}{11}-\frac{4}{11}$ **13.** $\frac{5}{6}-\frac{1}{6}$ **14.** $\frac{13}{14}-\frac{3}{14}$

15. $\frac{7}{4}-\frac{3}{4}$ **16.** $\frac{2}{9}+\frac{4}{9}$ **17.** $\frac{9}{7}+\frac{5}{7}$

18. $\frac{8}{15}-\frac{3}{15}$ **19.** $\frac{9}{18}+\frac{27}{18}$ **20.** $\frac{7}{25}+\frac{8}{25}$

Independent Practice

Now try these:
Lesson 3.4.3 additional questions — p442

Work out the sums and differences in Exercises 1–6.

1. $\frac{1}{4}+\frac{2}{4}$ **2.** $\frac{7}{3}-\frac{5}{3}$ **3.** $\frac{6}{14}+\frac{9}{14}$

4. $\frac{126}{3}-\frac{119}{3}$ **5.** $\frac{2}{11}+\frac{3}{11}$ **6.** $\frac{9}{35}+\frac{25}{35}$

Work out the sums and differences in Exercises 7–18. Give your answers in their simplest form.

7. $\frac{5}{42}+\frac{9}{42}$ **8.** $\frac{12}{66}+\frac{15}{66}$ **9.** $\frac{97}{20}-\frac{52}{20}$

10. $\frac{15}{15}-\frac{5}{15}$ **11.** $\frac{20}{18}-\frac{12}{18}$ **12.** $\frac{21}{7}-\frac{1}{7}$

13. $\frac{14}{12}-\frac{10}{12}$ **14.** $\frac{8}{9}+\frac{4}{9}$ **15.** $\frac{9}{21}+\frac{5}{21}$

16. $\frac{10}{2}+\frac{1}{2}-\frac{3}{2}$ **17.** $\frac{1}{3}+\frac{2}{3}+\frac{3}{3}$ **18.** $\frac{19}{8}-\frac{10}{8}+\frac{6}{8}$

Round Up

Adding and subtracting fractions with the same denominator is all about understanding the question. $\frac{2}{3}+\frac{5}{3}$ just means "two-thirds and then five-thirds more," so the result must be $\frac{7}{3}$.

The result won't always be in its simplest form, but you've already seen how to fix that.

Lesson 3.4.4

Fractions with Different Denominators

California Standards:

Number Sense 1.1
Compare and order **positive and negative fractions,** decimals, and mixed numbers and place them on a number line.

Number Sense 2.1
Solve problems involving addition, subtraction, multiplication, and division **of positive fractions** and explain why a particular operation was used for a given situation.

What it means for you:
In this Lesson, you'll learn about one way of adding and subtracting fractions with different denominators.

Key words:
- equivalent fraction
- denominator
- common multiple
- cross-multiply

After the last Lesson on adding fractions with the same denominator, this one is about adding fractions with different denominators. The idea here is to turn them both into fractions with the same denominator and then do exactly what you did in the last Lesson...

Every Pair of Numbers Has a Common Multiple

A **common multiple** of two numbers is a number that is a multiple of **both of them**. Take the numbers 2 and 3, for example:

Multiples of 2: 2, 4, **6**, 8, 10, **12**, 14...
Multiples of 3: 3, **6**, 9, **12**, 15...

The numbers 6 and 12 appear in both lists, so 6 and 12 are both **common multiples** of 2 and 3.

Example 1

Which of the following numbers are common multiples of 4 and 5?
8, 15, 20, 30, 32, 100

Solution

Common multiples will be multiples of 4 **and** multiples of 5.

From the list, the multiples of **4** are: 8, 20, 32, and 100.
From the list, the multiples of **5** are: 15, 20, 30, and 100.

So the **common multiples** of 4 and 5 are **20 and 100**.

Don't forget:
A multiple of a number is that number multiplied by an integer.

A quick way to find a common multiple of two numbers is to **multiply them together**. So $2 \times 3 = 6$ must be a common multiple of 2 and 3.

Example 2

Find a common multiple of 11 and 13.

Solution

One common multiple is given by $11 \times 13 = 143$.

Multiplying numbers together like this will **always** produce a common multiple. But it might not be the **smallest** common multiple.

Example 3

Find a common multiple of 12 and 24.

Solution

One common multiple is given by $12 \times 24 = 288$.

However, 24 is already a multiple of 12, so another possible answer is **24**.

Check it out:
You'll learn a method for finding the least common multiple in the next Lesson.

Guided Practice

Find a common multiple of each pair of numbers in Exercises 1–8. Try to find the smallest common multiples you can.

1. 1, 3 **2.** 5, 8 **3.** 10, 2 **4.** 6, 7
5. 11, 5 **6.** 3, 4 **7.** 5, 7 **8.** 8, 4

Common Multiples Make Common Denominators

Check it out:
When two fractions have the same denominator, you can say that they have a common denominator.

Look at the fraction sum $\frac{1}{3}+\frac{4}{7}$. This time, you can't just add the numerators because the two fractions have **different denominators**.

So you need to find two fractions with a **common denominator** — one fraction that is equivalent to $\frac{1}{3}$ and another that is equivalent to $\frac{4}{7}$.

This common denominator has to be a **common multiple** of 3 and 7.

Don't forget:
To find an equivalent fraction, multiply by a fraction equivalent to 1. See Lesson 3.4.1 for a reminder.

Example 4

Find fractions equivalent to $\frac{1}{3}$ and $\frac{4}{7}$ that share a common denominator.

Use your equivalent fractions to find $\frac{1}{3}+\frac{4}{7}$.

Solution

You need to find any common multiple of 3 and 7.
The easiest one to use is $3 \times 7 = 21$.

Now find fractions equivalent to $\frac{1}{3}$ and $\frac{4}{7}$ that have a denominator of 21.

$$\frac{1}{3}=\frac{1}{3}\times\frac{7}{7}=\frac{1\times7}{3\times7}=\frac{7}{21} \qquad \frac{4}{7}=\frac{4}{7}\times\frac{3}{3}=\frac{4\times3}{7\times3}=\frac{12}{21}$$

Then add these equivalent fractions:

$$\frac{1}{3}+\frac{4}{7}=\frac{7}{21}+\frac{12}{21}=\frac{7+12}{21}=\frac{19}{21}$$

Example 5

Find $\frac{1}{2}+\frac{1}{3}$.

Solution

First, make fractions equivalent to $\frac{1}{2}$ and $\frac{1}{3}$ that share a denominator.
Use $2 \times 3 = 6$ as the common denominator.

$$\frac{1}{2}=\frac{1}{2}\times\frac{3}{3}=\frac{3}{6} \qquad \frac{1}{3}=\frac{1}{3}\times\frac{2}{2}=\frac{2}{6}$$

Then add the new fractions: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{3+2}{6}=\frac{5}{6}$

Use exactly the same method for subtracting fractions.

Example 6

What is $\frac{3}{4}-\frac{1}{9}$?

Solution

Use a common denominator of $4 \times 9 = 36$. Your equivalent fractions are:

$$\frac{3}{4}\times\frac{9}{9}=\frac{27}{36} \qquad \frac{1}{9}\times\frac{4}{4}=\frac{4}{36}$$

So $\frac{3}{4}-\frac{1}{9}=\frac{27}{36}-\frac{4}{36}=\frac{23}{36}$.

Guided Practice

Don't forget:
"Finding a difference" means doing a subtraction.

Check it out:
The results of sums with different denominators may need simplifying just like sums with same denominators.

Work out the sums and differences in Exercises 9–17.
Give your answers in their simplest form.

9. $\frac{1}{6}+\frac{1}{5}$ **10.** $\frac{1}{5}-\frac{1}{6}$ **11.** $\frac{2}{3}+\frac{1}{4}$

12. $\frac{1}{9}+\frac{1}{3}$ **13.** $\frac{1}{3}-\frac{1}{10}$ **14.** $\frac{1}{2}-\frac{1}{4}$

15. $\frac{5}{6}+\frac{4}{20}$ **16.** $\frac{7}{9}+\frac{6}{11}$ **17.** $\frac{6}{13}-\frac{2}{9}$

A Common Denominator Allows Comparison

Check it out:
It's difficult to say which fraction is greater if the denominators are different.

You can compare two fractions with a **common denominator** by comparing their numerators.

Example 7

Which is greater: $\frac{7}{9}$ or $\frac{5}{6}$?

Solution

Find equivalent fractions with a common denominator.
Use a common denominator of $9 \times 6 = 54$. Then:

$$\frac{7}{9}\times\frac{6}{6}=\frac{42}{54} \qquad \frac{5}{6}\times\frac{9}{9}=\frac{45}{54}$$

These fractions have the same denominator, so compare the numerators.

$42 < 45$, so $\frac{42}{54}<\frac{45}{54}$. That means that $\frac{7}{9}<\frac{5}{6}$.

Check it out
You could use a common denominator of 18.
$\frac{7}{9}=\frac{14}{18}$ and $\frac{5}{6}=\frac{15}{18}$.

Check it out:

You'll use cross-multiplication in another context later in the book. Cross-multiplication is also used to solve proportions (as you'll see in Lesson 4.1.4).

Cross-multiplication is also a way to compare fractions. It gives exactly the same results as you saw in the previous example, but it's possibly slightly easier to remember.

Look at the fractions in the previous example again: $\frac{7}{9}$ and $\frac{5}{6}$.

Cross-multiplication involves multiplying the numerator of each fraction by the denominator of the other.

$$\mathbf{42} = 7 \times 6 \leftarrow \frac{7}{9} \quad \frac{5}{6} \rightarrow 9 \times 5 = \mathbf{45}$$

You can see that these two numbers are the same as the numerators in the fractions you found in Example 7.

$$\frac{7}{9} = \frac{7\times6}{9\times6} = \frac{\mathbf{42}}{54} \quad \text{and} \quad \frac{5}{6} = \frac{5\times9}{6\times9} = \frac{\mathbf{45}}{54}$$

So to **compare** two **fractions**, you can **cross-multiply**, and compare the results.

Don't forget:

Think of cross-multiplication as an easy way to remember which numbers to multiply together when you put fractions over a common denominator. It's not really a brand new method.

Example 8

Which is greater: $\frac{3}{5}$ or $\frac{2}{3}$?

Solution

Cross-multiply: $\mathbf{9} = 3 \times 3 \leftarrow \frac{3}{5} \quad \frac{2}{3} \rightarrow 5 \times 2 = \mathbf{10}$

These are the numerators of equivalent fractions, which both have a common denominator of $5 \times 3 = \mathbf{15}$.

So $\frac{3}{5} = \frac{9}{15}$. And $\frac{2}{3} = \frac{10}{15}$.

This means $\frac{2}{3} > \frac{3}{5}$.

Example 9

Which is greater: $\frac{6}{7}$ or $\frac{13}{15}$?

Solution

Cross-multiply: $\mathbf{90} = 6 \times 15 \leftarrow \frac{6}{7} \quad \frac{13}{15} \rightarrow 7 \times 13 = \mathbf{91}$

This means: $\frac{6}{7} = \frac{6\times15}{7\times15} = \frac{\mathbf{90}}{105}$ and $\frac{13}{15} = \frac{13\times7}{15\times7} = \frac{\mathbf{91}}{105}$

So $\frac{13}{15} > \frac{6}{7}$ or $\frac{6}{7} < \frac{13}{15}$.

Don't forget:

"$>$" means "is greater than."
"$<$" means "is less than."

Guided Practice

Say which of the two numbers given in each of Exercises 18–25 is greater.

18. $\frac{19}{10}, \frac{17}{20}$ **19.** $\frac{3}{5}, \frac{2}{3}$ **20.** $\frac{5}{9}, \frac{6}{11}$ **21.** $\frac{17}{13}, \frac{4}{3}$

22. $\frac{4}{15}, \frac{5}{19}$ **23.** $\frac{5}{7}, \frac{11}{15}$ **24.** $\frac{9}{21}, \frac{4}{9}$ **25.** $\frac{15}{9}, \frac{8}{5}$

Independent Practice

Find two common multiples of the numbers in Exercises 1–4.

1. 4 and 9 **2.** 3 and 12 **3.** 20 and 6 **4.** 7 and 1

5. Find a common multiple of 5 and 9 by multiplying these numbers together.

6. Write $\frac{1}{5}$ and $\frac{4}{9}$ as equivalent fractions, using your answer to Exercise 5 as a common denominator.

7. Using your answer to Exercise 6, find $\frac{1}{5}+\frac{4}{9}$.

Work out the sums and differences in Exercises 8–15. Give your answers in their simplest form.

8. $\frac{4}{3}+\frac{2}{6}$ **9.** $\frac{1}{5}+\frac{5}{9}$ **10.** $\frac{5}{7}+\frac{8}{5}$ **11.** $\frac{4}{3}-\frac{2}{6}$

12. $\frac{1}{2}+\frac{3}{8}$ **13.** $\frac{7}{8}-\frac{2}{3}$ **14.** $\frac{1}{6}-\frac{1}{9}$ **15.** $\frac{5}{6}-\frac{2}{3}$

Say which of the two numbers given in each of Exercises 16–19 is greater.

16. $\frac{17}{19}, \frac{8}{9}$ **17.** $\frac{6}{13}, \frac{9}{19}$ **18.** $\frac{5}{21}, \frac{2}{8}$ **19.** $\frac{10}{23}, \frac{3}{7}$

20. Andre and Amy both try to work out $1\frac{7}{8}+2\frac{3}{4}$, but come up with different answers. Identify who got it right and find the mistake that the other made.

Andre

$$1\frac{7}{8}+2\frac{3}{4}$$
$$=\frac{15}{8}+\frac{11}{4}$$
$$=\frac{15}{8}+\frac{11}{8}$$
$$=\frac{26}{8}$$
$$=\frac{13}{4}$$

Amy

$$1\frac{7}{8}+2\frac{3}{4}$$
$$=\frac{15}{8}+\frac{11}{4}$$
$$=\frac{15}{8}+\frac{22}{8}$$
$$=\frac{37}{8}$$

Now try these:
Lesson 3.4.4 additional questions — p442

Round Up

You can sidestep the problem of adding fractions with different denominators by turning them into fractions with the same denominator. Of course, you have to turn them into equivalent fractions, or you'll end up answering a different question entirely.

Lesson 3.4.5

Least Common Multiples

The last Lesson showed you how to add and subtract fractions with different denominators. This Lesson is about the same thing, but this time using the least common multiple. It's pretty similar to the method in the last Lesson, but it means you don't end up having to multiply really big numbers if you don't need to.

The Least Common Multiple Is Less Than the Others

You can always find a common multiple of two numbers by multiplying them together. But this method doesn't always give you the **least common multiple** (LCM).

One way to find the least common multiple is to compare lists of multiples and find the first number in both lists. Take 6 and 9, for example.

Some multiples of 6: 6, 12, **18**, 24, 30, 36, 42, 48...
Some multiples of 9: 9, **18**, 27, 36, 45, 54...

So the least common multiple of 6 and 9 is **18**.

Example 1

Find the least common multiple of 16 and 28.

Solution

Multiples of 16: 16, 32, 48, 64, 80, 96, **112**, 128...
Multiples of 28: 28, 56, 84, **112**...

So the least common multiple of 16 and 28 is **112.**

The least common multiple **isn't always** less than the result of multiplying the numbers together.

Example 2

Find the least common multiple of 9 and 7.

Solution

Multiples of 9: 9, 18, 27, 36, 45, 54, **63**...
Multiples of 7: 7, 14, 21, 28, 35, 42, 49, 56, **63**...

So the least common multiple of 9 and 7 is **63**.

Guided Practice

Find the least common multiple of the pairs of numbers given in Exercises 1–12.

1. 2, 3	**2.** 4, 3	**3.** 12, 9	**4.** 10, 4
5. 6, 8	**6.** 15, 35	**7.** 30, 45	**8.** 4, 12
9. 14, 21	**10.** 25, 30	**11.** 6, 7	**12.** 55, 33

California Standards:

Number Sense 2.1
Solve problems involving addition, subtraction, multiplication, and division **of positive fractions** and explain why a particular operation was used for a given situation.

Number Sense 2.4
Determine the least common multiple and the greatest common divisor **of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions** or to find the reduced form for a fraction).

What it means for you:

You'll learn how to find the lowest number that two numbers both divide into exactly. You'll also see how this can help you with adding and subtracting fractions.

Key words:

- least common multiple
- greatest common divisor

Check it out:

The least common multiple of 16 and 28 is 112. This is a lot less than their product, which is 16 × 28 = 448.

Check it out:

Notice how you don't have to continue either of these lists beyond 63.
This is because you know that 9 × 7 = 63 is a common multiple, so you're only using these lists to see if you can find one that's smaller.

Check it out:
Sometimes GCD is used for "greatest common divisor."

Prime Factorization Can Get the LCM More Quickly

You can find the **least common multiple** of two numbers by **multiplying them together**, then **dividing** by the **greatest common divisor**.

Example 3

Find the least common multiple of 12 and 44.

Solution

First use prime factorization to find the greatest common divisor.

$12 = 2 \times 2 \times 3$
$44 = 2 \times 2 \times 11$

Greatest common divisor $= 2 \times 2 = \mathbf{4}$.

So the least common multiple of 12 and 44 is: $12 \times 44 \div 4 = \mathbf{132}$

Don't forget:
For more information about prime factorization and finding the greatest common divisor, see Lesson 3.4.2.

Guided Practice

Use prime factorization to find the LCM of the pairs of numbers given in Exercises 13–20.

13. 6, 8 **14.** 9, 12 **15.** 4, 6 **16.** 10, 15
17. 39, 15 **18.** 34, 32 **19.** 60, 45 **20.** 27, 45

The LCM Is a Good Choice for Common Denominator

Remember, when you need to add or subtract fractions with different denominators, you must put both fractions over a common denominator. This means you need to find a **common multiple** of the denominators.

Using the **least common multiple** means that you can work with smaller numbers.

Don't forget:
See Lesson 3.4.4 to remind yourself about using common multiples when adding fractions.

Example 4

What is $\frac{1}{12} + \frac{1}{30}$?

Solution

- $12 = 2 \times 2 \times 3$ $\quad 30 = 2 \times 3 \times 5$
 So the greatest common divisor of 12 and $30 = 2 \times 3 = 6$.
- Least common multiple of 12 and 30 is $12 \times 30 \div 6 = \mathbf{60}$.
- Now find fractions equivalent to $\frac{1}{12}$ and $\frac{1}{30}$ which have a common denominator of 60.

$$\frac{1}{12} = \frac{1}{12} \times \frac{5}{5} = \frac{\mathbf{5}}{\mathbf{60}} \qquad \frac{1}{30} = \frac{1}{30} \times \frac{2}{2} = \frac{\mathbf{2}}{\mathbf{60}}$$

Now you can add the fractions together.

$$\frac{1}{12} + \frac{1}{30} = \frac{5}{60} + \frac{2}{60} = \frac{5+2}{60} = \frac{\mathbf{7}}{\mathbf{60}}$$

Check it out:
If you'd used a common multiple of $12 \times 30 = 360$, the numbers you'd need to use would be much bigger, and much easier to make mistakes with. And you'd definitely need to simplify your answer.

$$\frac{1}{12} + \frac{1}{30} = \frac{30}{360} + \frac{12}{360} = \frac{42}{360} = \frac{\mathbf{7}}{\mathbf{60}}$$

Guided Practice

Check it out:
Leave any fractions greater than 1 as improper fractions.

Work out the sums and differences in Exercises 21–29.
Give your answers in their simplest form.

21. $\frac{13}{10}+\frac{9}{15}$ **22.** $\frac{4}{9}+\frac{5}{6}$ **23.** $\frac{1}{6}+\frac{5}{10}$

24. $\frac{5}{6}-\frac{1}{8}$ **25.** $\frac{3}{15}+\frac{2}{20}$ **26.** $\frac{13}{42}+\frac{5}{14}$

27. $\frac{4}{6}-\frac{2}{8}$ **28.** $\frac{7}{24}-\frac{1}{16}$ **29.** $\frac{2}{9}-\frac{7}{33}$

Independent Practice

Now try these:
Lesson 3.4.5 additional questions — p442

Find the least common multiple of the pairs of numbers given in Exercises 1–12.

1. 2, 9 **2.** 5, 7 **3.** 19, 3 **4.** 6, 10
5. 5, 35 **6.** 15, 21 **7.** 6, 33 **8.** 35, 55
9. 30, 66 **10.** 42, 28 **11.** 39, 12 **12.** 24, 16

13. Find the prime factorizations of 6 and 4.
14. Calculate the LCM of 6 and 4
15. Write $\frac{5}{6}$ and $\frac{1}{4}$, using the LCM of 6 and 4 as a common denominator.
16. Find $\frac{5}{6}-\frac{1}{4}$

Check it out:
Leave any fractions greater than 1 as improper fractions.

Work out the sums in Exercises 17–31.
Give your answers in their simplest form.

17. $\frac{2}{3}-\frac{1}{4}$ **18.** $\frac{3}{4}-\frac{1}{2}$ **19.** $\frac{1}{9}+\frac{2}{7}$

20. $\frac{2}{10}+\frac{15}{25}$ **21.** $\frac{8}{9}-\frac{5}{6}$ **22.** $\frac{13}{12}-\frac{3}{4}$

23. $\frac{8}{11}+\frac{18}{44}$ **24.** $\frac{13}{15}+\frac{12}{18}$ **25.** $\frac{9}{10}-\frac{9}{14}$

26. $\frac{7}{75}+\frac{15}{35}$ **27.** $\frac{4}{27}-\frac{1}{45}$ **28.** $\frac{1}{24}+\frac{15}{56}$

29. $\frac{10}{49}+\frac{6}{21}$ **30.** $\frac{7}{9}+\frac{1}{30}$ **31.** $\frac{31}{42}-\frac{10}{56}$

Round Up

Pausing for a second to find the least common multiple before trying to add or subtract two fractions with different denominators is a good way to avoid having to deal with really big numbers. Smaller numbers also make simplifying at the end easier.

Lesson 3.4.6

Mixed Numbers and Word Questions

This Lesson combines two different parts — the first is about how to deal with mixed number sums and the second is about how to make sense of sums written in words.

Mixed Numbers Are Like Sums

Mixed numbers are **like sums**. $1\frac{3}{5}$ means "one and three-fifths," and so is the same as $1+\frac{3}{5}$. One way to deal with questions that use mixed numbers is just to replace all the mixed numbers with this kind of sum.

Example 1

What is $1\frac{3}{5}+2\frac{1}{5}$?

Solution

$$1\frac{3}{5}+2\frac{1}{5}=(1+\frac{3}{5})+(2+\frac{1}{5})$$
$$=1+2+\frac{3}{5}+\frac{1}{5}$$
$$=3+\frac{4}{5}$$
$$=3\frac{4}{5}$$

This method works quite well for **simple additions**. But there's a better way of doing it for more complicated sums and differences.

Mixed Number Sums Can Be Done with Fractions

A more reliable way of doing mixed number sums is to **convert all the mixed numbers to improper fractions** before you start. Then you can use the addition and subtraction methods covered in the previous Lessons.

Using improper fractions, the above example would become:

Example 2

What is $1\frac{3}{5}+2\frac{1}{5}$?

Solution

$$1\frac{3}{5}+2\frac{1}{5}=\frac{8}{5}+\frac{11}{5}=\frac{8+11}{5}=\frac{19}{5}=3\frac{4}{5}$$

California Standards:

Number Sense 2.1
Solve problems involving addition, subtraction, multiplication, and division **of positive fractions and explain why a particular operation was used for a given situation.**

Number Sense 2.4
Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

What it means for you:

You'll apply all of the skills you've learned so far in this Section to mixed number questions and questions written in words.

Key words

- mixed number
- improper fraction
- fact

Don't forget:

You can do additions in any order because of the commutative and associative properties of addition. See Lesson 2.3.1 to remind yourself about the commutative property of addition.

Don't forget:

To convert a mixed number to an improper fraction, multiply the whole number part by the denominator and add it to the numerator. See Lesson 3.1.2 for more details.

If the fraction parts of the mixed numbers have **different** denominators, you'll need to find a **common denominator**.

Example 3

Find $4\frac{3}{7}-1\frac{2}{3}$.

Solution

- First form improper fractions.

$$4\frac{3}{7}=\frac{(4\times7)+3}{7}=\frac{28+3}{7}=\frac{31}{7} \qquad 1\frac{2}{3}=\frac{3+2}{3}=\frac{5}{3}$$

- Now you need to find a common denominator, so you have to find a **common multiple** of 7 and 3. Use $7 \times 3 = 21$.

$$\frac{31}{7}=\frac{31}{7}\times\frac{3}{3}=\frac{93}{21} \qquad \frac{5}{3}=\frac{5}{3}\times\frac{7}{7}=\frac{35}{21}$$

- Finally, you can subtract the fractions.

$$4\frac{3}{7}-1\frac{2}{3}=\frac{93}{21}-\frac{35}{21}=\frac{93-35}{21}=\frac{58}{21}$$

Example 4

What is $4\frac{1}{4}+\frac{1}{3}$? Give your answer as a mixed number.

Solution

$4\frac{1}{4}=\frac{16+1}{4}=\frac{17}{4}$, so you can rewrite the sum as $4\frac{1}{4}+\frac{1}{3}=\frac{17}{4}+\frac{1}{3}$.

The **least common denominator** of 3 and 4 is **12**.

So $4\frac{1}{4}+\frac{1}{3}=\frac{17}{4}+\frac{1}{3}=\frac{51}{12}+\frac{4}{12}=\frac{55}{12}$

And as a **mixed number** this is $4\frac{7}{12}$.

Don't forget:
There's more information about improper fractions and mixed numbers in Lesson 3.1.2.

Guided Practice

Work out the sums in Exercises 1–12.

1. $1\frac{1}{3}+\frac{2}{3}$ **2.** $2\frac{1}{3}+3\frac{1}{3}$ **3.** $1\frac{2}{3}+6\frac{2}{3}$

4. $4\frac{1}{2}+5\frac{1}{9}$ **5.** $7\frac{3}{4}+1\frac{3}{5}$ **6.** $5\frac{2}{3}-\frac{1}{3}$

7. $8-3\frac{1}{3}$ **8.** $7\frac{2}{3}-5\frac{1}{2}$ **9.** $6\frac{4}{5}-6\frac{1}{6}$

10. $6\frac{1}{5}-3\frac{4}{5}$ **11.** $26\frac{19}{30}-2\frac{10}{12}$ **12.** $8\frac{11}{20}-3\frac{7}{16}$

Word Questions Can Be Broken into Facts

Some math questions look complicated but become easier if you **break them down into pieces**.

Example 5

Jose and Stephanie are climbing a mountain. Jose climbs up $1\frac{1}{4}$ miles. Stephanie climbs up $2\frac{1}{4}$ miles but then climbs back down $\frac{2}{3}$ of a mile. Who is further up the mountain?

Solution

First **separate out** the facts you've been given.

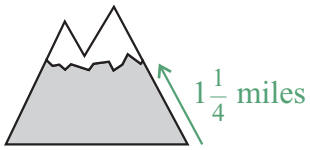
Jose and Stephanie are climbing a mountain. | Jose climbs up $1\frac{1}{4}$ miles. | Stephanie climbs up $2\frac{1}{4}$ miles | but then climbs back down $\frac{2}{3}$ of a mile. | Who is further up the mountain?

Then **put the facts together again as math**.

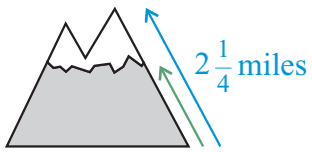
Jose and Stephanie are climbing a mountain.



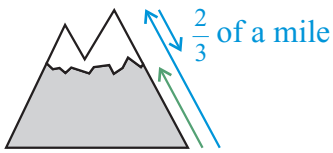
Jose climbs up $1\frac{1}{4}$ miles.



Stephanie climbs up $2\frac{1}{4}$ miles.

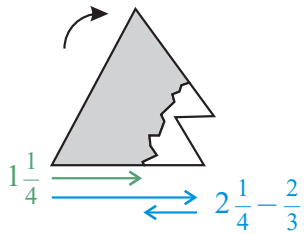


But then climbs back down $\frac{2}{3}$ of a mile.



Who is further up the mountain?

If you turn the mountain on its side then the edge is like a **number line**.



So the question is asking whether $2\frac{1}{4}-\frac{2}{3}$ is greater than $1\frac{1}{4}$.

$$2\frac{1}{4}-\frac{2}{3}=\frac{9}{4}-\frac{2}{3}=\frac{27}{12}-\frac{8}{12}=\frac{19}{12} \qquad 1\frac{1}{4}=\frac{5}{4}=\frac{15}{12}$$

$\frac{19}{12}$ is greater than $\frac{15}{12}$, so:

Stephanie is further up the mountain than Jose.

Don't forget:

When fractions have the same denominator, you can compare them by comparing their numerators. See Lesson 3.4.4 to remind yourself about comparing fractions.

Guided Practice

13. A recipe requires 2 cups of flour, 1 cup of milk, and $\frac{1}{4}$ cup of vegetable oil. How many cups of ingredients are needed in total?

14. Two years ago, Eric was $4\frac{1}{6}$ feet tall. Now, he is $5\frac{1}{3}$ feet tall. How much has he grown?

15. Cody has $2\frac{1}{2}$ pounds of bananas, but sells $1\frac{1}{4}$ pounds to Elizabeth. Elizabeth buys a further $\frac{2}{3}$ pounds from the store. How many pounds of bananas do Cody and Elizabeth end up with?

16. Carlos is doing a science experiment, and puts $\frac{1}{5}$ liter of water into a beaker containing $\frac{1}{8}$ liter of vinegar. He mixes them together and puts $\frac{1}{24}$ liter into a test tube. How much is left in the beaker?

Independent Practice

Work out the sums in Exercises 1–9.
Give your answers as mixed numbers.

1. $1\frac{1}{2}-\frac{1}{2}$ **2.** $1\frac{1}{2}+\frac{1}{2}$ **3.** $2\frac{1}{2}+3\frac{2}{3}$

4. $5\frac{4}{6}+2\frac{1}{9}$ **5.** $3\frac{1}{3}-1\frac{8}{21}$ **6.** $7\frac{1}{4}-6\frac{17}{18}$

7. $12\frac{1}{6}-11\frac{6}{15}$ **8.** $19\frac{1}{16}-10\frac{5}{8}$ **9.** $12\frac{1}{11}+16\frac{3}{17}$

10. Several students are having a race. Each contestant has to run 100 meters then cycle 1000 meters. Their times are shown below.
How long did each student take to finish, and who finished first?

	Karina	T.J.	Emily	Kyle
Running	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{1}{5}$
Cycling	$3\frac{2}{3}$	$2\frac{9}{10}$	$2\frac{3}{5}$	$3\frac{2}{9}$

(all times measured in minutes)

11. Laura and Juan are at a water fountain. Laura drinks $\frac{1}{8}$ gallon of water, and Juan drinks $\frac{3}{10}$ gallon. Laura is still thirsty, so she drinks $\frac{1}{5}$ gallon of fruit juice. Who drank more? How much more did that person drink?

Now try these:
Lesson 3.4.6 additional questions — p443

Round Up

Sometimes it can be difficult to make sense of long word problems, but just remember the key steps — break up the question and separate all the facts from one another. Then take each one individually and figure out what it means in math.

Section 3.5 introduction — an exploration into:

Percents with a Double Number Line

Percents can seem complicated. This is because they are an alternative way to describe an amount. For example, suppose you have 20 apples and then eat half of them. You've eaten 10 apples, but this is the same as 50%. One way to see the link is to use a double number line.

A double number line shows **percents** next to actual **values**. You can use it to **estimate** solutions.

Example

Use a double number line to estimate 30% of 40.

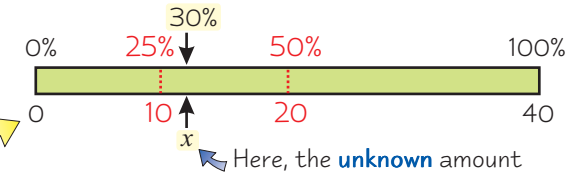
Solution

Draw a double number line.
Label one side of the number line from **0% to 100%**, and the other side from **0 to 40**.



You can use some easy-to-find points on the double number line to **estimate** the **value** of x.

- **50%** is **halfway** between **0% and 100%**.
 Its value is **half of 40**, which is **20**.
- **25%** is **halfway** between **0% and 50%**.
 Its value is **half of 20**, which is **10**.



30% is a bit more than 25%.
You can see that 30% is **between 10 and 20, but closer to 10**.
In fact, it looks like 30% of 40 **is about 12**.

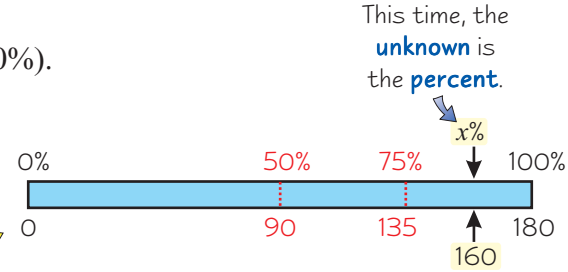
Example

160 is what percent of 180? Use a double number line to find an estimate.

Solution

Draw a double number line, and mark on **180** (= 100%).
Now mark in some other important percents.

- **50%** is **halfway** between **0% and 100%**.
 Its value is half of 180, which is **90**.
- **75%** is **halfway** between **50% and 100%**.
 Its value is halfway between 90 and 180, or **135**.



You can see that x is **between 75% and 100%**. In fact, it looks **about 85-90%**.

Exercises

1. Draw a double number line to estimate the solution to these percent problems.

a. What is 40% of 80?
b. What is 60% of 250?
c. 105 is what percent of 150?
d. 64 is 20% of what number?

Round Up

This method will only give you an estimate of the answer — it probably won't be exact. But estimates can be really useful for checking your work, or getting a rough idea of the answer.

Lesson 3.5.1

Section 3.5
Fractions and Percents

Percents are another way of writing fractions, and they're used a lot in real life. You might have percents as test scores, or on a sale poster in a store. In this Lesson, you'll see how to turn a fraction into a percent.

California Standard:

Number Sense 1.4

Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

What it means for you:

You'll learn about percents as another way to write fractions.

Key words:

- percent
- fraction
- numerator
- denominator
- equivalent fraction
- greatest common divisor

Don't forget:

The numerator is on the top of the fraction.
The denominator is on the bottom.
For example:

$\frac{3}{4}$ ⇦ numerator
⇦ denominator

Percents Are a Different Way to Write Fractions

Percents are a way to write a **fraction** as a **single number**.

One percent is the same as the fraction $\frac{1}{100}$.

Percents are written using the symbol **%**.

For example: **23 percent**
is written **23%**
and means $\frac{23}{100}$.

Example 1

An auto dealership has 100 cars for sale. 32 of the cars are red. What percent of the cars are red?

Solution

The fraction of the cars that are red is $\frac{32}{100}$.

Written as a percent, that is **32%**.

Of course, fractions don't always come in **hundredths**.
To turn a fraction into a percent, you can use **equivalent fractions**.

Example 2

Write $\frac{4}{5}$ as a percent.

Solution

Rewrite $\frac{4}{5}$ in hundredths: $\frac{4}{5} = \frac{8}{10} = \frac{80}{100} = 80\%$

You can find the same answer using algebra. You need to **multiply** $\frac{4}{5}$ by a fraction equivalent to 1 to make the **denominator** 100.

Multiply $\frac{4}{5}$ by $\frac{n}{n}$.

Denominator of the result = $5 \times n = 100$

So $n = 100 \div 5 = 20$

So to make the denominator **100**, multiply $\frac{4}{5}$ by $\frac{20}{20}$:

$$\frac{4}{5} = \frac{4}{5} \times \frac{20}{20} = \frac{4 \times 20}{5 \times 20} = \frac{80}{100} = 80\%$$

Guided Practice

In Exercises 1–6, write each fraction as a percent.

1. $\frac{65}{100}$ **2.** $\frac{1}{4}$ **3.** $\frac{21}{50}$ **4.** $\frac{3}{25}$ **5.** $\frac{11}{20}$ **6.** $\frac{9}{25}$

7. A bag contains 50 marbles, 27 of which are green.
What percent of the marbles are green?

8. There are 20 students riding the bus, and 14 of them are girls.
What percent of the students are girls?

There Is Another Way to Convert Fractions to Percents

There is an alternative method for converting a fraction to a percent.
You can just **do the division** shown by the fraction, then **multiply by 100**.

Example 3

Write $\frac{1}{8}$ as a percent.

Solution

Do the division shown by the fraction (here, **1 ÷ 8**),
and then **multiply by 100**.

So $1 \div 8 =$ **0.125**.
Then multiply by 100 to get: $0.125 \times 100 = 12.5$, so $0.125 =$ **12.5%**.

Whether you **convert** a fraction to **hundredths**, or **do the division** and then **multiply by 100**, the answer should come out the same.

Example 4

Main Street Park has 64 trees. Of those, 28 are pine trees.
What percent of the trees are pine?

Solution

Write the answer as a fraction: $\frac{28}{64}$ of the trees are pine.

1. You can convert this to hundredths by multiplying by $\frac{n}{n}$: $\frac{28}{64} \times \frac{n}{n} = \frac{?}{100}$

Looking at the denominators, you can see that $64 \times n = 100$.
This means $n = 100 \div 64 =$ **1.5625**.

So $\frac{28}{64} = \frac{28}{64} \times \frac{1.5625}{1.5625} = \frac{28 \times 1.5625}{64 \times 1.5625} = \frac{43.75}{100} =$ **43.75%**.

2. Or you can find: $28 \div 64 =$ **0.4375**.
Then multiply by 100: $0.4375 \times 100 = 43.75$, so $0.4375 =$ **43.75%**.

Don't forget:
You need to understand both methods, but you can use whichever method you prefer.

Guided Practice

In Exercises 9–12, write each fraction as a percent.

9. $\frac{929}{1000}$ **10.** $\frac{23}{200}$ **11.** $\frac{17}{40}$ **12.** $\frac{3}{8}$

You Can Turn a Percent into a Fraction

It's straightforward to turn a percent into a **fraction**.
Don't forget to rewrite the fraction in its **simplest form**.

Don't forget:
For a reminder about finding the greatest common divisor and the simplest form of a fraction, see Lesson 3.4.2.

Example 5

Write 40% as a fraction in simplest terms.

Solution

40% is the same as $\frac{40}{100}$.

The greatest common divisor of 40 and 100 is 20.

$\frac{40}{100} = \frac{2 \times \cancel{20}}{5 \times \cancel{20}} = \frac{2}{5}$, so $40\% = \frac{2}{5}$

Guided Practice

In Exercises 13–20, write the percent as a fraction in its simplest form.

13. 39% **14.** 38% **15.** 50% **16.** 75%
17. 5% **18.** 98% **19.** 35% **20.** 16%

Independent Practice

In Exercises 1–6, write each fraction as a percent.

1. $\frac{77}{100}$ **2.** $\frac{1}{10}$ **3.** $\frac{16}{25}$ **4.** $\frac{7}{8}$ **5.** $\frac{23}{40}$ **6.** $\frac{111}{500}$

Write each of the following percents as a fraction in its simplest form.

7. 9% **8.** 20% **9.** 70% **10.** 86% **11.** 95% **12.** 92%

13. Cora buys 16 apples at the store. She buys 4 red and 12 green apples. What percent of Cora's apples are green?

14. DaMarcus recorded the weather on 80 days last winter. It snowed on 4 of those days. On what percent of the days did it snow?

15. Out of 250 grade 6 students, 150 have brown eyes.
What percent of the students have brown eyes?

16. Seth is practicing his baseball skills. He hits 880 out of 1000 pitches. What percent of the balls does he hit?

Miguel and Claire go to the park to play tennis.
There are 25 tennis courts at the park.
When Miguel and Claire get there, 5 courts are available.
17. What percent of the tennis courts are available?
18. What percent of the tennis courts are not available?

Now try these:
Lesson 3.5.1 additional questions — p443

Round Up

Percents are often easier to work with than fractions. For example, comparing percents is usually more straightforward than comparing fractions. A percent is one number, so there aren't any numerators and denominators to worry about.

Lesson 3.5.2

Percents and Decimals

California Standard:
Number Sense 1.4
Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

What it means for you:
You'll see how to rewrite percents as decimals, and decimals as percents.

Key words:
- percent
- decimal
- hundredth
- order

You can rewrite fractions as percents to make them easier to compare and use. Percents can be converted to decimals too, and this sometimes helps when you're using them in calculations.

Percents Can Be Turned into Decimals

You've seen that **percents** are basically the same as **hundredths**.
You've also seen that **one-hundredth** is the same as the decimal **0.01**.

So $\frac{1}{100} = 1\% = 0.01$.

This means that you can **rewrite** any percent as a decimal by **dividing by 100**.

Example 1

Write 5% and 10% as decimals.

Solution

$5\% = \frac{5}{100} = 5 \div 100 = \mathbf{0.05}$ $\quad 10\% = \frac{10}{100} = 10 \div 100 = \mathbf{0.1}$

Don't forget:
You can leave out any extra zeros on the right-hand side of a decimal number.
So 0.10 is written as 0.1.

Guided Practice

In Exercises 1–8, write each percent as a decimal.

1. 18%	**2.** 99%	**3.** 9%	**4.** 2%
5. 20%	**6.** 50%	**7.** 23.2%	**8.** 61.78%

Decimals Can Be Turned into Percents

It's also straightforward to turn a **decimal** into a **percent**.

Example 2

Write 0.04 and 0.82 as percents.

Solution

0.04 = 4 hundredths $= \frac{4}{100} = \mathbf{4\%}$ $\quad$ 0.82 = 82 hundredths $= \frac{82}{100} = \mathbf{82\%}$

Notice that this time, you've **multiplied by 100**.

Guided Practice

In Exercises 9–16, write each decimal as a percent.

9. 0.35	**10.** 0.81	**11.** 0.9	**12.** 0.06
13. 0.6	**14.** 1.0	**15.** 0.195	**16.** 0.001

You Should Become Familiar with Common Percents

The table below shows some **fractions**, **decimals**, and **percents** that are **equal** to each other. The table will be useful for solving some questions. You might already be familiar with some of these values.

Check it out:
This table will help you to answer some of the questions you'll see in the next few Lessons.

Table of Common Percents and their Equivalents		
Fraction	**Decimal**	**Percent**
$\frac{1}{100}$	0.01	1%
$\frac{1}{10}$	0.1	10%
$\frac{1}{5}$	0.2	20%
$\frac{1}{4}$	0.25	25%
$\frac{1}{2}$	0.5	50%
$\frac{3}{4}$	0.75	75%
1	1	100%

Example 3

There are 32 students in Mr. Baker's art class, and 8 of them are in the school play. What percent of Mr. Baker's art students are in the play?

Solution

First, you can represent this as a fraction: $\frac{8}{32}$

Writing in simplest terms, $\frac{8}{32} = \frac{1}{4}$

Looking at the table above, you can see that $\frac{1}{4}$ is equal to 25%.

So, **25%** of Mr. Baker's students are in the play.

Guided Practice

In Exercises 17–20, use the table above to find the right percent.

17. In a class of 30 students, 3 have pet cats. What percent of the class have cats?

18. Angela has 20 pens. 5 of them are blue. What percent of Angela's pens are blue?

19. Tony travels 15 miles to get home. He goes 12 miles by bus, then walks 3 miles. What percent of the journey does Tony walk?

20. Donna buys 13 apples and 13 oranges.
What percent of Donna's fruit is apples?

Percents Make It Easier to Place Numbers in Order

Sometimes you can't tell right away which of two numbers is **greater**. They might be fractions with different denominators, or you might have one fraction and one decimal.

Changing everything into **percents** makes things easier to put in order.

Example 4

Order the following from least to greatest:

$\frac{2}{5}$ 0.5 35%

Solution

The table on the previous page tells you that $\frac{1}{5} = 20\%$.

So $\frac{2}{5} = 2 \times \frac{1}{5} = 2 \times 20\% = \mathbf{40\%}$.

And 0.5 = 50 hundredths = $\frac{50}{100} = \mathbf{50\%}$.

Now you can see that $35\% < 40\% < 50\%$.

So $\mathbf{35\% < \frac{2}{5} < 0.5}$.

Check it out:
You may find decimals easier to understand than percents. If you do, it's okay to answer this type of question by turning all the numbers into decimals instead.

Guided Practice

In Exercises 21–24, order each set of numbers from least to greatest.

21. 72%, $\frac{7}{10}$, 0.82

22. $\frac{1}{20}$, 3%, 0.09

23. 45%, 0.4, $\frac{12}{25}$, $\frac{23}{50}$

24. 0.9, $\frac{17}{20}$, 88%, $\frac{3}{4}$

Independent Practice

In Exercises 1–4, write each percent as a decimal.

1. 54% **2.** 3% **3.** 70% **4.** 2.5%

In Exercises 5–8, write each decimal as a percent.

5. 0.84 **6.** 0.3 **7.** 0.08 **8.** 0.172

In Exercises 9–12, order each set of numbers from least to greatest.

9. 0.2, 18%, $\frac{1}{4}$

10. $\frac{11}{20}$, 54%, 0.5

11. $\frac{3}{5}$, $\frac{33}{50}$, 0.65, 63%

12. 39%, $\frac{9}{25}$, $\frac{21}{50}$, 0.35

13. A box contains 50 red pens and 150 blue pens. What percent of the pens in the box are blue?

14. There are 4 red and 16 blue marbles in a bag. What percent of the marbles are blue?

Now try these:
Lesson 3.5.2 additional questions — p443

Round Up

The table showing the common percents, fractions, and decimals is really useful. It will help you answer percent questions more quickly. You should try to memorize it if you can.

Lesson 3.5.3

Percents of Numbers

California Standard:
Number Sense 1.4
Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

What it means for you:
You'll learn how to find a certain percent of a number.

Key words:
- percent
- multiply
- fraction
- decimal

Check it out:
You can find the fraction and decimal equivalents of 25% in the table of common percents and equivalents from Lesson 3.5.2.

You saw earlier in this Chapter how to multiply whole numbers by fractions. A percent is another way to write a fraction, so you can multiply whole numbers by percents too. This is called finding a percent of a number.

Multiply to Find a Percent of a Number

You've already practiced multiplying **fractions** by **integers**.

Finding a percent of a number is similar because, for example,

"1% of 250" means $1\% \times 250 = 0.01 \times 250 = \frac{1}{100} \times 250$.

So it usually helps to convert the percent to a **fraction** or **decimal** first.

Example 1

What is 25% of 140?

Solution

There are a number of ways you can solve this problem.

Use a fraction: Written as a fraction, $25\% = \frac{1}{4}$.

$$\frac{1}{4} \times 140 = \frac{140}{4}$$

$$= \mathbf{35}$$

Use a decimal: Written as a decimal, $25\% = \mathbf{0.25}$.

$$0.25 \times 140 = \mathbf{35}$$

Find 1% first: 1% of $140 = 140 \div 100 = \mathbf{1.4}$

So 25% of $140 = 25 \times (1\%$ of $140)$
$= 25 \times 1.4$
$= \mathbf{35}$

Guided Practice

In Exercises 1–12, find the following percents:

1. 10% of 100	**2.** 2% of 50	**3.** 20% of 3000
4. 80% of 50	**5.** 12% of 750	**6.** 44% of 800
7. 95% of 80	**8.** 4% of 950	**9.** 35% of 80
10. 15% of 240	**11.** 85% of 130	**12.** 29% of 250

Real-life Situations Can Involve Percents of Numbers

You can often find a **percent of a number** in real-world situations.

Check it out:
You can solve this type of question using either fractions or decimals.
You can normally solve the questions more quickly using the decimal method with a calculator. But you should get some practice with the fraction method too.

Example 2

In a survey of 200 people leaving a movie theater, 65% said that they enjoyed the movie. How many people in the survey enjoyed the movie?

Solution

The problem is asking you to find 65% of 200.

65% of 200 = 0.65 × 200 = **130**

So **130** of the people in the survey enjoyed the movie.

Guided Practice

13. Carla has 12 friends at school, and 50% of them ride on the same bus as she does. How many friends ride on the same bus as Carla?

14. A mixture of 30 liters of soap and water contains 10% soap. How many liters of the mixture is soap?

15. A school football team has won 70% of their last 60 games. How many of those games did they win?

16. A parking lot contains 150 cars, and 18% of them are blue. How many blue cars are in the parking lot?

Some Questions Work the Other Way Around

If you know a **percent of a number** you can find the **number itself**.

Example 3

25% of the animals in a pet store are dogs. There are 10 dogs. How many animals are there in the store?

Solution

25% of the number of animals is 10.

You can write this as:	25% × the number of animals = 10
Or:	0.25 × the number of animals = 10
So:	the number of animals = 10 ÷ 0.25 = **40**.

Or you can do this using algebra...

Call the number of animals n. Since 25% = $\frac{1}{4}$, you know $\frac{1}{4} \times n = 10$.

This means $n = 10 \div \frac{1}{4} = 10 \times 4 =$ **40**. There are **40 animals**.

Check it out:
You can use this algebraic approach using decimals:
25% = 0.25, and
0.25 × n = 10, so
n = 10 ÷ 0.25 = 40.

Guided Practice

Solve Exercises 17–21, using either fractions or decimals.

17. 50% of a number is 13. What is the number?

18. Nancy has 27 green marbles. If 75% of Nancy's marbles are green, how many marbles does Nancy have in total?

19. 4% of the trees in a park are oak trees. If there are 7 oak trees in the park, how many trees are there altogether?

20. 25% of a number is 30. What is 75% of the number?

21. In a survey of favorite animals, 2% of people answered ducks, and 10% said elephants. 14 people said their favorite animals are ducks. How many said their favorite animals are elephants?

Independent Practice

In Exercises 1–4, find the following percents:

1. 70% of 20 **2.** 3% of 600

3. 25% of 1200 **4.** 45% of 220

5. Darnell has 200 baseball cards. He gives 20% of his collection to his friend Kenny. How many cards does Darnell give to Kenny?

6. There are 25 pet cats living on 7th Street. Mrs. Delgado owns 16% of them. How many cats does Mrs. Delgado own?

7. Mr. Jones plants 80 daffodil bulbs in his garden. 85% of them grow successfully. How many daffodil plants does Mr. Jones end up with?

8. A zoo has 425 animals, and 4% of them are penguins. How many penguins does the zoo have?

9. The school soccer team scored 75 goals last season. Luisa scored 36% of the goals. How many goals did Luisa score last season?

Use either fractions or decimals to solve Exercises 10–14.

10. 10% of what number is 12?

11. 220 is 80% of what number?

12. 50% of a number is 8. Calculate 25% of the number.

13. In the class elections, 20% of the students voted for Elena. If Elena received 54 votes, how many students voted in total?

14. Phoebe is counting the vehicles that pass her house. 10% of them are motorcycles and 50% are cars. If Phoebe counts 6 motorcycles, how many cars does she count?

Now try these:
Lesson 3.5.3 additional questions — p444

Round Up

It's really important that you understand this Lesson. In the rest of this Section, you'll learn about circle graphs, percent increase and decrease, and simple interest. All of these will involve finding percents of numbers.

Lesson 3.5.4

Circle Graphs and Percents

California Standard:
Number Sense 1.4
Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

What it means for you:
You'll use circle graphs to compare different percents.

Key words:
- percent
- circle graph
- 100%
- compare

You'll have seen circle graphs before in earlier grades. They're often used to compare different percents. Circle graphs are useful because they show clearly how the size of one group relates to another.

100% is Equal to the Whole Amount

100% of something is **all of it**.

It's important to remember this when you're looking at questions about percents.

Example 1

Yesenia has a number of marbles. 65% of the marbles are red.
The rest are all blue. What percent of Yesenia's marbles are blue?

Solution

All the marbles are either red or blue.

So (percent of red marbles) + (percent of blue marbles) = all marbles

$$\begin{aligned} 65\% + \text{percent of blue marbles} &= 100\% \\ \text{percent of blue marbles} &= 100\% - 65\% \\ &= 35\% \end{aligned}$$

So **35%** of Yesenia's marbles are blue.

Guided Practice

1. 47% of students in a school are boys. What percent are girls?

2. A store took a survey of customers' ages.
29% said they were under 15. What percent were aged 15 or over?

3. 18% of a class of 6th graders have a pet dog.
What percent of the class don't have a dog?

4. A bag contains a number of colored counters. All the counters are either blue, green, or yellow. 30% of the counters are blue, and 40% of the counters are green. What percent of the counters are yellow?

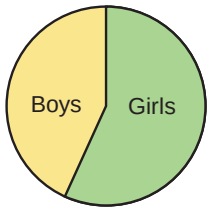
5. Mrs. Goldman's garden has only red, white, and blue flowers.
37% of the flowers are red, and 39% of the flowers are white.
What percent of the flowers are blue?

Action	27%
Comedy	41%
Romance	?

6. Visitors to a movie theater were asked which of 3 types of movie they liked best.
This table shows the results.
What is the missing percent?

Circle Graphs Are Often Divided into Percents

Circle graphs show how a total splits into different parts.



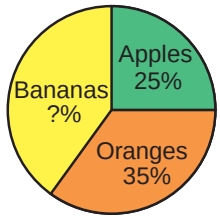
This graph represents a math class split into boys and girls.
The **whole circle** represents the **whole class**.
The **two sections** represent the **boys** and the **girls**.

The girls section is **larger**.
This means the class has **more girls** than boys.

When a circle graph shows **percents**, the whole circle represents **100%**.
A section that **represents** a certain **percent** fills that percent of the circle.

Example 2

This circle graph shows the results of a survey to find which out of apples, bananas, and oranges students liked best.



What percent of the students like bananas best?

Solution

The whole circle represents **100%**, so the **total value** of all the sections must be 100%.

Call the percent of students who like bananas best b%.

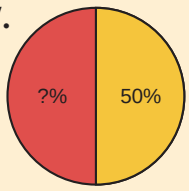
Then
$25 + 35 + b = 100$ **The total of all sections is 100%**
$60 + b = 100$ **25 + 35 = 60**
$b = 100 - 60 = 40$ **Subtract 60 from both sides**

So **40%** of the students like bananas best.

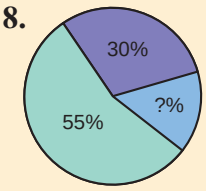
Guided Practice

In Exercises 7–10, find the missing value in each circle graph.

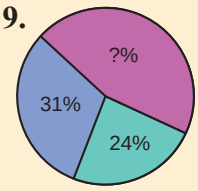
7.



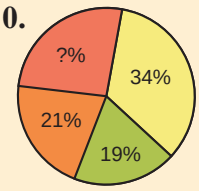
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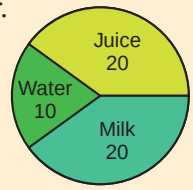
9.



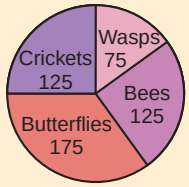
10.



A survey asked people which of three drinks they prefer. This graph shows the number of people who said they prefer each drink.



11. How many people answered the survey?
12. What fraction of these people preferred each drink?
13. What percent of these people preferred each drink?



This graph shows the number of insects a scientist counted for a study.

14. How many insects were there in total?
15. What percent of each type of insect were there?

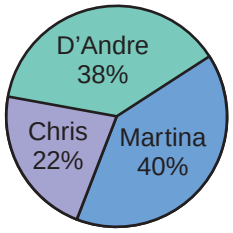
Don't forget:
In Exercises 11–15, the numbers on the circle graphs aren't percentages, but actual quantities.

You Can Turn Percents on Circle Graphs into Numbers

If you know how many **units** make up the full **100%** of a circle graph, then you can work out how many each section represents.

Example 3

Chris, Martina, and D'Andre each ran for student body president. A total of 150 students voted in the election, and the outcome of the election is shown in the circle graph.
How many students voted for Martina?



Solution

The circle graph tells you that 40% of the students voted for Martina. You know that the whole circle graph represents 150 students.

So the number of students who voted for Martina is

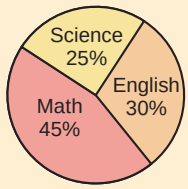
$$40\% \text{ of } 150 = 0.4 \times 150 = 60$$

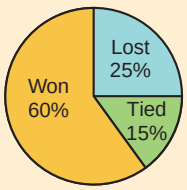
A total of **60 students** voted for Martina.

Guided Practice

Use the graphs provided to answer Exercises 16–19.

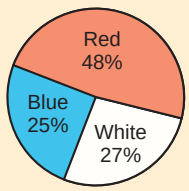
16. This graph shows the results of a survey of students' favorite school subjects. 80 students were surveyed. Find the number of students that liked each subject best.

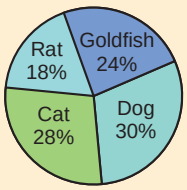




17. The percents of games won, lost, and tied by a soccer team out of their last 40 games are shown on this graph. How many of the games did they win, lose, and tie?

18. This graph shows the percent of each color of balloon at a party. There were a total of 200 balloons at the party. Find how many of each color of balloon there were at the party.





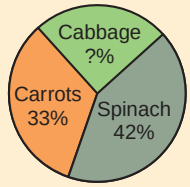
19. 300 people were surveyed as to which of 4 types of animals they liked the best. This graph shows the results. How many people in the survey liked each type of animal?

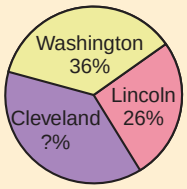
Independent Practice

1. A group of sixth-grade students each go to one of 5 clubs after school on Tuesdays.
The table shows what percent of the students go to each club.
What percent of the students go to the reading club? Explain your answer.

Acting Club	15%
Reading Club	?
Debate Club	10%
Music Club	40%
Recycling Club	10%

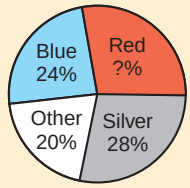
2. A group of people were asked which of 3 vegetables they like best. This graph shows the results. Calculate the missing percent.
3. 300 people were questioned. Find the number of people who said they like each vegetable.

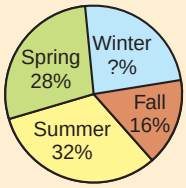




4. A town held a vote to decide who to build a statue of. The results are shown on this graph.
Find the missing percent.
5. 250 people voted. Figure out how many people voted for each choice.

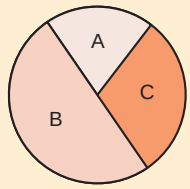
6. This graph shows the colors of cars in the school parking lot. What percent of the cars are red?
7. If there are 175 cars in the lot, how many of each color are there?



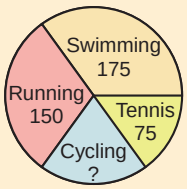


8. This graph shows the results of a survey asking people about their favorite season. Figure out the missing percent.
9. The survey questioned 325 people. How many said they like each season the best?

Mr. Benson has 300 acres of land on his farm. He grows corn, lettuce, and tomatoes. There are 60 acres of lettuce and 90 acres of tomatoes.
10. How many acres of corn are there?
The graph shows how the land on the farm is divided between the three crops.
11. Which section of the graph represents each crop? Explain your answer.
12. What percent of the land is used for each crop?



Now try these:
Lesson 3.5.4 additional questions — p444



500 sixth-graders were asked which of four sports they liked best. This graph shows the results.
13. How many students said they liked cycling best?
14. Write down what fraction of the sixth-graders preferred each sport.
15. What percent of the students preferred each sport?

Round Up

Circle graphs are useful for giving out information. People who don't know much about math can still understand what it means when one part of the circle is bigger than another.

Lesson 3.5.5

Percent Increase

California Standard:
Number Sense 1.4
Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and **tips**.

What it means for you:
You'll solve problems where percents are used to describe how much a number has increased.

Key words:
- percent
- increase
- add
- tips
- sales tax

Don't forget:
You convert a percent to a decimal by dividing by 100.

Don't forget:
Check that all your answers are reasonable.

One real-world use of percents is to describe how certain numbers have changed. People say something has increased or decreased by a certain percent. In this Lesson, you'll learn about percent increase.

You Can Increase a Number by a Certain Percent

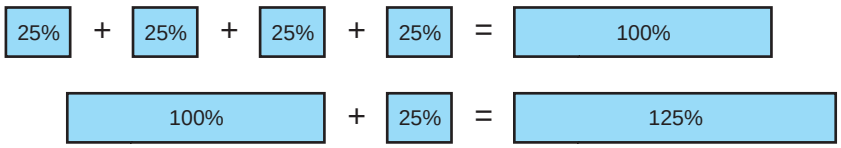
When you **add** to a number, **percent increase** is a way to describe how much it's changed. The percent increase is the **amount** of the increase, written as a **percent** of the **original number**.

Example 1

Increase 200 by 25%.

Solution

"Increase by 25%" means "add on 25% of the original number."



The amount of the increase is

$$200 \times 25\% = 200 \times 0.25 = \mathbf{50}$$

So to increase 200 by 25%, **add 50**.

$$200 + 50 = \mathbf{250}$$

There's another, slightly quicker way to find this answer.
The answer is 125% of the original amount, so you want

$$200 \times 125\% = 200 \times 1.25 = \mathbf{250}$$

Guided Practice

In Exercises 1–6, calculate the total after the increase.

1. Increase 100 by 12%.
2. Increase 50 by 20%.
3. Increase 160 by 25%.
4. Increase 60 by 5%.
5. Increase 185 by 60%.
6. Increase 275 by 30%.

Tips Are Calculated Using Percents

When people eat at a restaurant, they often **tip** the person who waited on them, or served their food. People often work out how much to tip by **finding a percent** of the bill.

Check it out:
In real life, people often estimate the right percent to tip, or round the amount up. But in math you should work out the exact tip, and only round to the nearest cent, unless an exercise tells you to do otherwise.

Example 2

Victor goes out to lunch. The bill comes to $14.00, and Victor wants to leave a 20% tip for the waiter. How much tip should he leave?

Solution

Victor wants to leave a **20%** tip, so he should **multiply** the dollar amount by **20%**.

$$20\% \times \$14.00 = 0.2 \times \$14.00 = \mathbf{\$2.80}$$

So Victor should leave a tip of **$2.80**.

Example 3

Aisha buys a meal in a restaurant. The meal costs $25.00, and Aisha decides to leave an 18% tip. How much does Aisha pay in total?

Solution

The **amount** of the increase is $\$25.00 \times 18\%$
$= \$25.00 \times 0.18 = \mathbf{\$4.50}$.

So the total Aisha pays is $\$25.00 + \$4.50 = \mathbf{\$29.50}$.

Guided Practice

7. Rick is at a cafe. The bill for his meal is $10.
If Rick leaves a 20% tip, how much does he pay in total?

8. Dolores is buying breakfast. It costs $21, and Dolores leaves a 19% tip. How much does she leave as a tip?

9. The Jackson family goes out for a meal. The bill comes to $150, and Mrs. Jackson leaves a 15% tip. How much does she leave as a tip?

10. Wayne's restaurant bill is $120. If Wayne leaves a tip of 25%, how much does he pay in total?

11. Francine is a waitress. A customer adds a 22% tip to his $42 bill. How much of a tip does he leave for Francine?

12. The cost of Phil's meal is $64. Phil wants to leave a tip of 18%. How much should he pay in total?

You Can Figure Out the Percent Increase

If you know the **original number**, and you know what it's been **increased** to, you can find the percent increase.

Example 4

50 is increased to 58. What is the percent increase?

Solution

You could solve this by **guess and check**:

Try 10%	$50 \times 110\% = 50 \times 1.1 =$ **55**	**— too small**
Try 20%	$50 \times 120\% = 50 \times 1.2 =$ **60**	**— too large**
Try 15%	$50 \times 115\% = 50 \times 1.15 =$ **57.5**	**— too small**
Try 16%	$50 \times 116\% = 50 \times 1.16 =$ **58**	**— correct**

Increasing 50 to 58 is an increase of **16%**.

You can also use **algebra**:

original amount × percent increase = amount of increase

The original amount is **50**, and the amount of the increase is $58 - 50 =$ **8**.

So $50 \times p = 8$ **Call the percent increase p**

$p = 8 \div 50$ **Divide both sides by 50**

$= 0.16 = 16\%$

So increasing 50 to 58 is an increase of **16%**.

Example 5

Lucy's restaurant bill came to \$200. She paid \$238.
What percent tip did Lucy leave?

Solution

The **amount** of the increase is \$238 – \$200 = **\$38**

So $200 \times p = 38$ **Call the percent increase p**

$p = 38 \div 200$ **Divide both sides by 200**

$= 0.19 = 19\%$

So Lucy left a tip of **19%**.

Guided Practice

In Exercises 13–18, work out the percent increase.

13. 100 increased to 142.
14. 25 increased to 30.
15. 160 increased to 184.
16. 36 increased to 63.
17. 45 increased to 49.5.
18. 260 increased to 377.

19. Wanda buys lunch for $15.00 plus a tip.
If the total she pays is $18.00, what percent tip has she left?

20. Guillermo pays $41.30 for his meal, including a tip.
The original bill was $35. What percent did Guillermo tip?

21. Diana is paying for a meal. The bill is $25.00, and Diana wants to tip at least 18%. If she pays $30, has she left enough as a tip?

22. AJ spends $59.40 on a pair of sneakers. The price without sales tax is $55. What percent sales tax does AJ pay?

Check it out:
Sales tax is calculated in the same way as tips – as a percent of the price before the tax is added.

Independent Practice

In Exercises 1–4, find the total after the increase.

1. Increase 50 by 40%.
2. Increase 220 by 5%.
3. Increase 70 by 17%.
4. Increase 56 by 65%.

In Exercises 5–8, work out the percent increase.

5. 110 increased to 121.
6. 60 increased to 75.
7. 350 increased to 427.
8. 12 increased to 14.4.

9. Corey's bill for lunch is $25. He decides to tip 16%.
How much does Corey leave as a tip?

10. After sales tax, Mr. Johnson paid $535 for a dishwasher that cost $500 before sales tax. What percent did he pay in sales tax?

11. Ms. Connor gave the hairstylist $50. Her haircut cost $40.
What percent of the bill did Ms. Connor leave as a tip for the stylist?

12. Justin is buying a football. The cost before sales tax is $30.
If the sales tax is 8%, how much does Justin pay in total?

13. With sales tax, the cost of Gavin's new hat is $21.50. If the sales tax is 7.5%, what is the cost of the hat, not including tax?

14. Mrs. Gomez takes a cab to the airport. The fare is $85.47, and she pays $100. What percent tip does Mrs. Gomez give the cab driver?

15. Zina, Adriana, and Raymond all buy dinner in the same restaurant. Zina's bill is $28.50, and she leaves a 20% tip. Adriana's bill is $30. She leaves a tip of 18%. Raymond adds a 22% tip to his bill of $27. How much do they each pay in total? Who pays the most?

Now try these:
Lesson 3.5.5 additional questions — p444

Round Up

Percent increases are one-half of the story. Next Lesson, you'll learn about percent decreases. The math you'll need for that is almost identical to what you've learned in this Lesson.

Lesson 3.5.6

Percent Decrease

California Standard:

Number Sense 1.4

Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

What it means for you:

You'll solve problems where percents are used to describe how much a number has decreased.

Key words:

- percent
- decrease
- subtract
- discount

Check it out:

The way you calculate percent decrease is just like what you saw last Lesson for percent increase. The only difference is that this time the original number is getting smaller, not bigger.

In the last Lesson, you saw how to use percents to describe the change when you add to a number. When you subtract from a number, you can describe the change in a similar way. This is percent decrease.

You Can Decrease a Number by a Certain Percent

You can use **percents** to describe what happens when you **decrease** a number. **Percent decrease** is the **amount** of change, described as a **percent** of the **original number**.

Example 1

Decrease 200 by 25%.

Solution

The **amount** of the decrease is

$$200 \times 25\% = 200 \times 0.25 = \mathbf{50}$$

So to **decrease** 200 by 25%, **take away 50**.

$$200 - 50 = \mathbf{150}$$

You can also say that if you take away **25%**, then you're left with $100\% - 25\% = \mathbf{75\%}$ of the original number.

$$\begin{aligned} 75\% \text{ of } 200 &= 75\% \times 200 \\ &= 0.75 \times 200 = \mathbf{150} \end{aligned}$$

Guided Practice

In Exercises 1–6, calculate the total after the decrease.

1. Decrease 100 by 15%.
2. Decrease 150 by 50%.
3. Decrease 80 by 20%.
4. Decrease 132 by 25%.
5. Decrease 300 by 35%.
6. Decrease 125 by 6%.

7. A number is decreased by 44%. What percent of the number is left?

8. A number is decreased by 75%.
What fraction of the number is left?

Percent Decrease Is Used to Describe Discounts

Percent decrease is often used to describe the **discount** when the **price** of an item that's for sale is **lowered**.

Example 2

A store discounts a coat by 20%. The coat originally cost $90. What is the discounted price of the coat?

Solution

The **amount** of the decrease is

$$\$90 \times 20\% = \$90 \times 0.2 = \mathbf{\$18}$$

So the **discounted price** is

$$\$90 - \$18 = \mathbf{\$72}$$

Check it out:
You could also work this out using the fact that if the discount is 20%, the new price is 100% – 20% = 80% of the old price. You can work this out as:
$90 × 80% = $90 × 0.8 = $72.

Guided Practice

9. Kitty buys some pens. They were originally priced at $3, but now there is a 50% discount. How much does Kitty pay for the pens?

10. This table shows some items on sale at a sporting goods store.

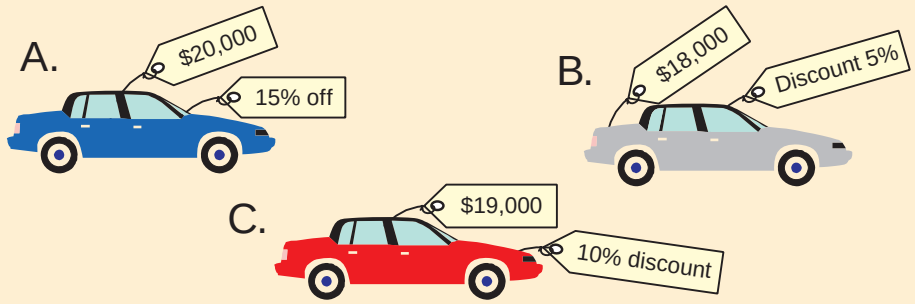
Calculate the sale price of each item. Which item has the lowest price after the discount?

Item	Original price	Percent discount
Basketball	$50	18%
Hockey stick	$60	30%
Skateboard	$70	40%

11. A travel agent offers Mrs. Winters a discount of 17% on a vacation. If the original price was $3500, how much does Mrs. Winters pay?

12. Juliana buys a clock that originally cost $24, but now has a 5% discount. How much does Juliana pay for the clock?

Three cars are shown below with the original price and the discount available for each car.
13. Find the price of each car after the discount.
14. Which car has the lowest price after the discount?



You Can Figure Out the Percent Decrease

Finding the percent **decrease** is like finding the percent **increase**.

Example 3

If 45 is decreased to 27, what is the percent decrease?

Solution

original amount × percent decrease = amount of decrease

The original amount is **45**. The amount of the decrease is 45 – 27 = **18**.

So $45 \times p = 18$ **Call the percent decrease p**

$p = 18 \div 45$ **Divide both sides by 45**

$= 0.4 = 40\%$

So decreasing 45 to 27 is a decrease of **40%**.

Example 4

A table originally priced at \$210 has been discounted to \$189. What is the percent discount?

Solution

The **amount** of the decrease is

\$210 – \$189 = **\$21**

Now find what **percent** of \$210 is equal to \$21. Call that percent p.

$210 \times p = 21$

$p = 21 \div 210 = 0.1 = 10\%$

So the discount is **10%**.

Guided Practice

In Exercises 15–20, work out the percent decrease.

15. 200 decreased to 180.
16. 120 decreased to 90.
17. 55 decreased to 11.
18. 160 decreased to 152.
19. 500 decreased to 60.
20. 325 decreased to 286.

21. Denzel goes to the bookstore. He buys a book for \$15 that has been discounted from \$20. What is the percent discount?

22. Brian buys a sweater that has been reduced from \$20 to \$11. What is the percent discount?

Independent Practice

1. The stationery store is selling everything for 40% off. Michael buys a pencil sharpener that originally cost $1. How much does he pay?

2. Mykelti buys a pair of pants in a sale. They have been discounted from $22 to $11. What percent discount is this?

3. Lauren buys a pair of boots which originally cost $68. The store gives a discount of 25%. How much does Lauren pay?

4. Ms. Dominguez is buying a set of golf clubs. They have been reduced from $400 to $320. What is the percent discount?

5. Donny's new bike has a 35% discount on it. If the original price was $260, what is the discounted price?

6. Tina wants to buy a new scarf. She sees one that is reduced by 40% from $16. What is the new price?

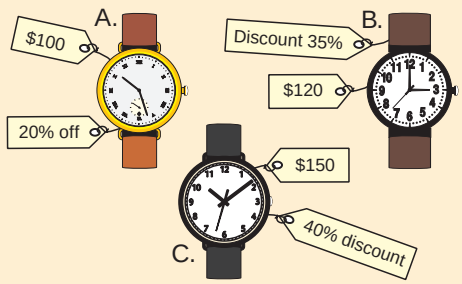
7. Kia is buying a couch for her apartment. The furniture store is offering a discount on one couch from $800 to $700. What percent discount is this?

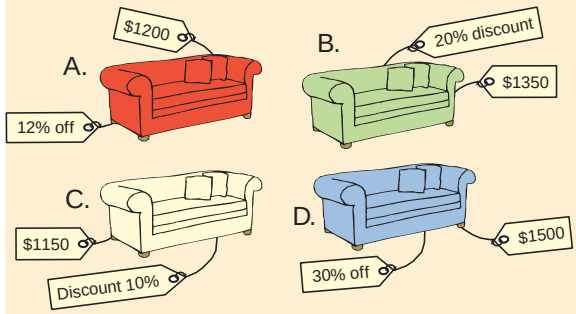
Luther is eating at a restaurant. His bill comes to $50, and he uses a coupon to get a discount of 20%.

8. What is the cost of the meal after the discount has been taken off?

9. Luther decides to leave a tip of 25% of the bill, after the discount has been taken off. How much does he pay in total?

10. These three watches have all been reduced. The labels show the original prices and the percent discount. Find the discounted price of each watch, and say which costs the least after the discount.





11. The labels on these couches show the original prices and the percent discounts.
Find the discounted price of each couch, and say which has the lowest price after the discount.

Now try these:
Lesson 3.5.6 additional questions — p445

Round Up

The topics of percent increase and percent decrease are very similar. Once you've got the hang of one, the other should make sense too. Practice plenty of both types of questions until you're comfortable with them.

Lesson 3.5.7

Simple Interest

California Standard:
Number Sense 1.4
Calculate given percentages of quantities and solve problems involving discounts at sales, **interest earned**, and tips.

What it means for you:
You'll learn about simple interest, which is a common use of percents.

Key words:
- percent
- simple interest
- principal
- rate

Banks pay people when they put their money into savings accounts. But if someone borrows money, they usually have to pay back some extra on top of what they borrow. These are both examples of interest. In this Lesson, you'll learn about one type of interest.

Interest Is Added to Loans or Savings

If you **borrow** money, you usually have to repay **more** than you borrowed. The **extra** money you **pay** is **interest**.

If you **save** money in the bank, the bank **pays you** to save your money with them. The **extra** money you **receive** is also called **interest**.

Example 1

Pablo borrowed $1000 to buy a car. One year later, he repaid $1050. How much interest did Pablo pay?

Solution

The interest is the **difference** between what Pablo **borrowed** and what he **paid back**.

So Pablo paid $1050 – $1000 = **$50** interest.

When you're dealing with interest, the time is usually given in years.

The original amount that is borrowed or saved is called the **principal**.

Guided Practice

Sara puts $200 in a savings account.
After 1 year, the interest has increased it to $210.
1. How much interest did Sara's money earn?
2. What was the principal?

Matthew borrows $500. He repays the loan over 24 months.
The total he pays back is $560.
3. What was the principal?
4. How many years did it take Matthew to repay his loan?
5. How much interest did Matthew pay?

Omar puts $400 into a savings account. He leaves the money in the account for 48 months. The account pays $25 interest each year.
6. What is the principal?
7. How many years did Omar leave his money in the account for?
8. How much interest did he earn in this time?

Simple Interest Is One Type of Interest

Simple interest is when the interest paid **each year** is a certain **percent** of the principal. The percent that's added is called the **interest rate**.

Example 2

Lakeesha puts $300 into an account with a simple interest rate of 8%.
How much interest will the bank pay in one year?

Solution

The amount of simple interest the bank will add on each year is 8% of the principal of $300 that Lakeesha has paid in.

$$8\% \text{ of } \$300 = 0.08 \times 300 = 24$$

The bank will pay **$24** into the account in one year.

The amount of **simple interest** paid in a certain length of time
= (the amount of interest paid in **one year**) × (the **number of years**)
= (the **principal**) × (the interest **rate**) × (the number of **years**)

Check it out:
Annual (as in annual interest rate) is another word for yearly.

You can rewrite this as a mathematical formula:

$$I = Prt$$

Where I = **interest** (the amount of interest paid)

P = **principal** (the original amount borrowed or saved)

r = **rate** (the annual interest rate, written as a decimal)

t = **time** (the time in years you're calculating interest for)

Example 3

Patrick borrows $350 at a simple interest rate of 4%.
He repays the loan after 6 months.
How much interest will he pay?

Solution

You can use $I = Prt$: $P = \$350$, $r = 4\% = 0.04$, $t = 6$ months $= 0.5$ years

So $I = Prt = 350 \times 0.04 \times 0.5 = 7$

Patrick will pay **$7** in simple interest at the end of 6 months.

Don't forget:
For a reminder of how to work with expressions, see Chapter 2.

Guided Practice

Don't forget:
To work out the total someone owes or the balance in their savings account, add the interest to the principal.

Check it out:
When you use $I = Prt$, remember that t is always the number of years.
So if you're dealing with a period of 6 months, put that into the equation as half a year, or 0.5 years.

Exercises 9–16 are about simple interest. Use the formula $I = Prt$ to answer them.

9. Christina puts \$100 in a 5% savings account for 3 years.
How much interest does she earn?

10. Dan borrows \$125 for 2 years. The interest rate is 20%.
How much will he have to pay in total?

11. Bonnie saves \$600 in an account with an interest rate of 4%.
After 4 years, how much interest will she have earned?

12. Hernan puts \$350 into a bank account paying 6% interest.
What will be the total in his account in 2 years?

13. Sally borrows \$400, with an annual interest rate of 8%.
She repays the loan after 3 months. How much interest does she pay?

14. Keisha puts \$200 into a savings account. How much interest will she earn in 6 months if the interest rate is 10%?

15. Steve takes out a loan of \$150. The interest rate is 12%.
How much does Steve owe in total after 4 months?

16. Tyreese saves \$240. If his account pays 15% interest, how much interest will he receive in 3 years?

Rearrange $I = Prt$ to Solve Other Problems

By rearranging $I = Prt$, you can solve other problems involving simple interest.

Example 4

Dawn paid \$200 into a savings account. After 3 years, she had earned \$30 simple interest. What is the yearly interest rate?

Solution
You can use $I = Prt$.

You know that $I = 30$, $P = 200$, and $t = 3$. Now you want to find r.

$I = Prt$	**Write out the formula**
$30 = 200 \times r \times 3$	**Put in values for I, P, and t**
$30 = 600r$	**$200 \times 3 = 600$**
$r = \frac{30}{600} = 0.05 = 5\%$	**Divide both sides by 600**

The annual interest rate on Dawn's savings account is **5%**.

Guided Practice

Use the formula $I = Prt$ to answer Exercises 17–22.

17. Joanna puts $100 into a savings account. In 4 years, she receives $28 interest. What is the interest rate on Joanna's account?

18. Cleavon saves $200, with an interest rate of 6%. How long does it take Cleavon to earn $60 interest?

19. Simon borrows some money with a 10% interest rate. After 3 years, he has repaid the loan with $36 interest. How much did Simon borrow?

20. Rosa borrows $50 at an interest rate of 12%. She repays the loan with $12 interest. How long does it take Rosa to repay the loan?

21. Kate's savings account has a 10% interest rate. In 6 months, she receives $55 interest. How much money did Kate put in the account?

22. John puts $250 in his savings account. In 2 years, he earns $25 interest. What is the interest rate?

Independent Practice

Exercises 1–8 are all about simple interest.

1. Winnie borrows $700, at an interest rate of 13%. How much interest does she owe after 2 years?

2. Jorge puts $1700 in a savings account. If the interest rate is 5%, how much interest does Jorge get in 3 years?

3. Aleesha saves $1200. Her bank pays her an interest rate of 7%. How much interest does Aleesha receive in 4 months?

4. Martin borrows $245 for 6 months. How much interest does he pay, if the interest rate is 10%?

5. Chloe borrows $300, and pays 15% interest on the loan. She owes $180 interest after how many years?

6. In 1 year, Aurelia earns $14 interest on her savings. If the interest rate is 8%, how much did she save?

7. Caleb's savings of $80 earn $36 interest over 5 years. What is the interest rate on Caleb's account?

8. Austin put $260 in a savings account at a simple interest rate of 8.5% for 4 years. Joe put $325 in a savings account at a simple interest rate of 8% for 4 years. Who earned the most in interest at the end of 4 years?

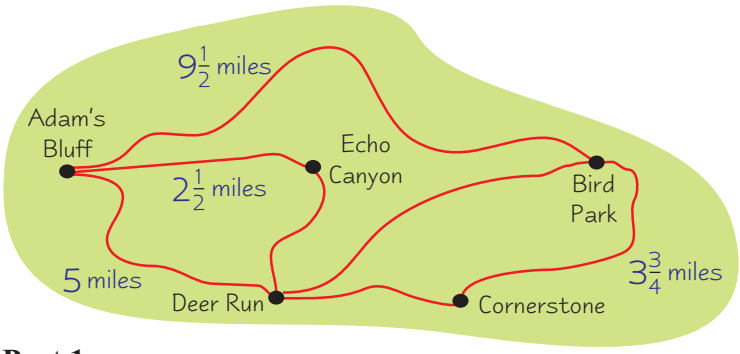
Now try these:
Lesson 3.5.7 additional questions — p445

Round Up

In real life, it's useful to be able to understand interest. People who understand interest rates can make sure they get a good deal when they want to save or borrow money.
Banks often use a more complicated type of interest, called compound interest.
You won't be learning about that in grade 6, but you might see it in math lessons in later grades.

Chapter 3 Investigation
Wildlife Trails

You're a park ranger responsible for the trails at a nature park.
A colleague has suggested organizing a Park Fun Run, and you need to help her plan it.
Use the map and the additional information below.



Additional Information:
- The distance from Deer Run to Bird Park is half a mile shorter than the distance from Bird Park to Cornerstone.
- The distance from Deer Run to Cornerstone is one-third of the distance from Bird Park to Cornerstone.
- The distance from Echo Canyon to Deer Run is 5% shorter than the distance from Echo Canyon to Adam's Bluff.

Part 1:
How long are the trails from:

- Bird Park to Deer Run?
- Deer Run to Cornerstone?
- Echo Canyon to Deer Run?

Part 2:
Your colleague wants the Park Fun Run to cover all the trails **once**.

- Is this possible? Where could it start and finish?
- How long would the Park Fun Run be?

Extensions

- Mark wants to hike from Adam's Bluff to Bird Park along the shortest route. What route should he take? What is the length of this route in miles?
- A new trail is being planned between Echo Canyon and Bird Park. The trail will be exactly $2\frac{1}{2}$ times the length of the trail from Deer Run to Cornerstone.
 (i) What is the length, in miles, of the new trail?
 (ii) With this new trail, will it still be possible to organize a Park Fun Run that covers all the trails **once**? If possible, describe a suitable new route.

Open-ended Extension

- The organizers of the Park Fun Run need to provide drinks for the runners. Drinks stations can be placed at the above locations, and should be as evenly spaced as possible. Find the best locations for them (assuming no new trails have yet been built).
- An additional trail from Deer Run to Cornerstone has been suggested. How will this affect the route of the Park Fun Run?

Round Up

Being able to use fractions is an incredibly useful skill. Buy two drinks for a dollar and a half each, and you'll be adding fractions to work out the total.

Chapter 4

Ratio, Proportion and Rate

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Section 4.1 introduction — an exploration into:

Billboard Ratios

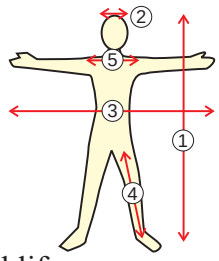
Photographs don't usually show objects at their actual size. However, some things are the same on a photograph as they are in real life. For example, if a person's shoulders are actually 3 times wider than their head, then this will also be true in a photo. This is what you are going to explore...

You need a recent **photograph** of yourself.

With a partner measure:
(i) **yourself**, and (ii) the **picture** of yourself.

The measurements you need are shown in this table.
Make a copy of the table to record your measurements.

Body measurement	Picture	Real life	
height	12	160	①
head width	1.5	20	②
arm span			③
leg length			④
shoulder width			⑤



Use **centimeters** for all your measurements. Also, make the measurements in real life and on the picture in the **same way** (for example, measure your arm span fingertip to fingertip).

Example

Find the ratios of the measurements on the picture to the real-life measurements.

Solution

Add an extra column to your table. To work out the ratios, **divide** each measurement on the **picture** by the corresponding **real-life** measurement.

Ratio picture ÷ real life
0.075
= 12 ÷ 160

Exercises

1. What do you notice about your ratios? How do you explain this?

Example

Your picture is used on a billboard poster. In this picture you are 10 meters (1000 cm) tall.
Use ratios to determine the width of your head on the poster.

Solution

The ratio of **height on the billboard** to **real-life height** is: $\frac{\text{billboard height}}{\text{real-life height}} = \frac{1000}{160} = 6.25$

This is the same for **all** measurements, so: $\frac{\text{billboard head width}}{\text{real-life head width}} = \frac{\text{billboard head width}}{20} = 6.25$

This tells you that on the billboard, your head would be **6.25 × 20 cm = 125 cm wide**.

Exercises

2. Calculate each of your measurements for a billboard on which you will be 10 meters tall.

3. Calculate your measurements if your picture were put on a postage stamp.
On the postage stamp, you will be only 1.5 cm tall.

Round Up

This was all about ratios. You saw that the ratio of measurements on a picture to measurements in real life is the same, no matter what you measure. Very useful... as you'll see later in the Section.

Lesson 4.1.1

Section 4.1 Ratios

California Standard:

Number Sense 1.2

Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) **to show the relative sizes of two quantities**, using appropriate notations (**a/b, a to b, a:b**).

What it means for you:

You'll learn what ratios are and the different ways you can represent them.

Key words:

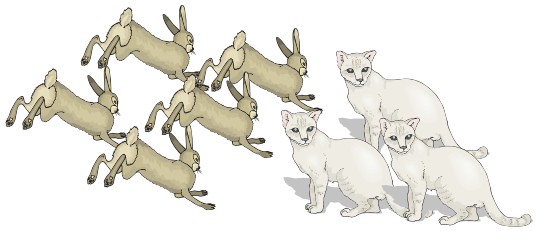
- ratio
- fraction

You've probably come across ratios before — for example, making orange drink by diluting about one part concentrated drink with about four parts water. This way of stating how the amount of one thing compares with the amount of another thing is called a ratio.

Ratios Are a Way of Comparing Amounts of Things

A **ratio** is the amount of one thing **compared** with the amount of another thing.

In the diagram, the **ratio of rabbits to cats is 5 to 3**. This is because there are **5 rabbits** but only **3 cats**.



Example 1

What is the ratio of carrots to hamsters in the picture below? And what is the ratio of hamsters to carrots?



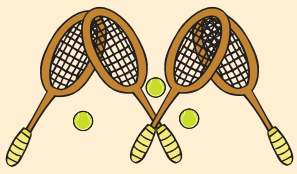
Solution

Count the number of carrots. There are **3 carrots**.
Count the number of hamsters. There are **2 hamsters**.

Make sure you get the **order** of the numbers right.
The ratio of **carrots to hamsters** has the number of **carrots** first.
The ratio of carrots to hamsters is 3 to 2.
In the same way, the **ratio of hamsters to carrots is 2 to 3.**

Guided Practice

1. What is the ratio of tennis rackets to tennis balls in the picture opposite?



2. What is the ratio of tennis balls to tennis rackets in the picture?

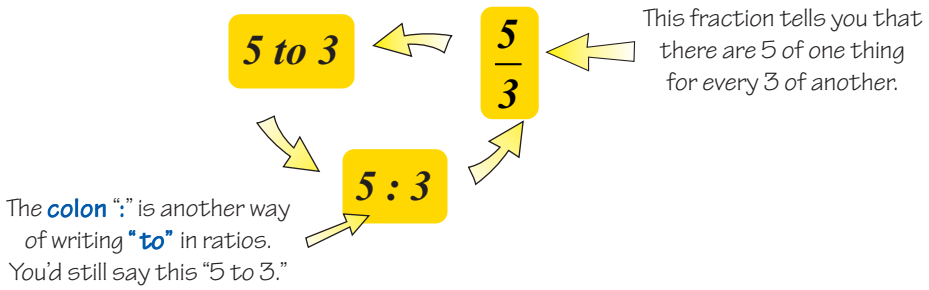
3. There are ten people on a basketball court and two baskets. What is the ratio of people to baskets on the court?

There Are Three Ways of Writing a Ratio

The previous examples used words like "5 to 3" to write a ratio.
But there are other ways to write ratios, and they all mean the **same thing**.

There are **three** ways you need to know about.

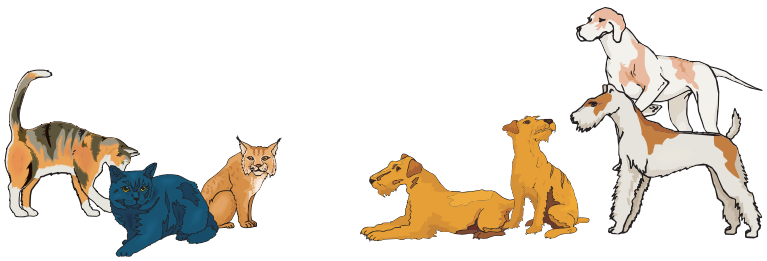
Check it out:
A ratio is really just another name for a fraction.



Example 2

The ratio of dogs to cats in the picture below is 4 to 3.
How else could you write this ratio?

Write the ratio of cats to dogs in three different ways.



Solution

There are three ways of writing ratios.

You could also write the ratio of dogs to cats as **4 : 3** or $\frac{4}{3}$.

The ratio of cats to dogs is **3 to 4**, or **3 : 4**, or $\frac{3}{4}$.

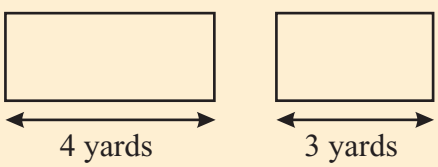
Check it out:
Notice that forming a ratio basically involves dividing one number by another.
Writing a ratio as a fraction shows this most clearly.

Guided Practice

4. The ratio of cars to people on a street is 1 : 2.
How else can you represent this ratio?

5. The ratio of boys to girls in a class is $\frac{2}{3}$.
Express the ratio of girls to boys in three different ways.

6. What is the ratio of the width of the first rectangle shown below to the second?



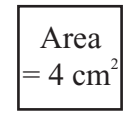
Ratios Don't Have Any Units

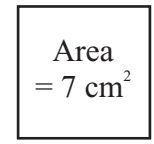
Ratios **don't have any units**. A ratio is just one number compared with another, such as **"3 to 2."**

Usually, ratios compare things measured in the **same units**.
Then, the units **cancel out** when you divide the two quantities.

Example 3

What is the ratio of the area of the big square on the right to the smaller one?





Solution

To form the ratio of the areas, **divide** one by the other.

Ratio of big area to small area $= \frac{7\ \cancel{\text{cm}^2}}{4\ \cancel{\text{cm}^2}} = \frac{7}{4}$

Don't forget:
Ratios can be written as fractions.

But if the units are different, the **description** of what the ratio shows is usually a bit more **detailed**.

Example 4

A walker hikes 7 miles in 3 hours. What is the ratio of the distance walked in miles to the time taken in hours?

Solution

Ratio of the **distance walked in miles** to the **time taken in hours** $= \frac{7}{3}$.

Distance walked in miles

Time taken in hours

Don't forget:
See also Lesson 4.4.1 — where rates are covered.

Guided Practice

Jess makes purple paint using 4 cans of blue paint to 1 can of red.
It takes her 3 hours to paint her bedroom, and she uses all her paint.

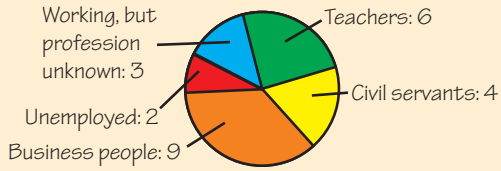
7. What is the ratio of cans of blue paint to cans of red paint?
8. What is the ratio of cans of paint used to the time taken in hours?

Independent Practice

In Exercises 1–4, write the ratio shown in two equivalent ways.

1. 7 : 6 **2.** 3 to 8 **3.** $\frac{11}{7}$ **4.** 7 : 11

Exercises 5–7 are based on the the circle graph on the right. It shows the professions of 24 former pupils of a school.



5. What is the ratio of people working in business to civil servants?
6. What is the ratio of unemployed people to people in business?
7. What is the ratio of people who are working and whose professions are known, to working people whose professions are unknown?

Now try these:
Lesson 4.1.1 additional questions — p445

Round Up

You've now met all of the basics of ratios. Next, you'll see how to use ratios in real life and how to simplify them to make your calculations easier.

Lesson 4.1.2

Equivalent Ratios

California Standard:

Number Sense 1.2

Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) **to show the relative sizes of two quantities**, using appropriate notations (**a/b, a to b, a:b**).

What it means for you:

You'll learn how to form different ratios with the same value, and to find the simplest form of a ratio.

Key words:

- simplifying
- equivalent

In real life, you might have to scale ratios up or down — like when you change a cake recipe from serving 4 people to serving 8 people. In this Lesson you'll learn how to find equivalent ratios that allow you to scale quantities. It's very similar to making equivalent fractions, so some of this might seem familiar to you.

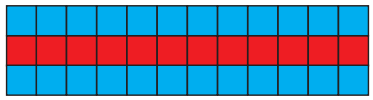
Equivalent Ratios Show the Same Thing

Fractions like $\frac{2}{4}$ and $\frac{1}{2}$ are different, but they represent the same thing. They're called equivalent fractions.

Equivalent ratios are the same kind of thing.

Example 1

What is the ratio of red squares to blue squares in the pattern on the right?



Solution

There are different answers to this question.

Altogether, there are 12 red squares and 24 blue squares, so one answer is **12 : 24**.

But you can also say that the ratio of red to blue squares is **1 : 2**. This is because there is 1 red square for every 2 blue squares.

The ratios 12 : 24 and 1 : 2 are **equivalent ratios**.

Don't forget:

You simplify ratios in exactly the same way as you simplify fractions.
For example,

$$\frac{12}{24} = \frac{1\times12}{2\times12} = \frac{1}{2}.$$

Look back to Lessons 3.4.1 and 3.4.2 for more information.

If you have two equivalent ratios, then you can say that the one that uses smaller numbers is **simpler**. So **1 : 2** and **12 : 24** are **equivalent** ratios, but 1 : 2 is **simpler**.

In fact, 1 : 2 is in its **simplest form**, since no other equivalent ratio uses smaller numbers.

To simplify a ratio, divide both parts by the same number.
For example, to simplify 12 : 24, **divide** both sides by **12** to get **1 : 2**.

Example 2

Andrew has a bag of marbles. The bag contains 4 blue and 14 black marbles. What is the ratio of blue to black marbles in its simplest form?

Solution

There are **4 blue** marbles, and **14 black** marbles. So the ratio of blue to black marbles is **4 : 14**.



But you can simplify the ratio 4 : 14 to **2 : 7**.

Guided Practice

Simplify the ratios in Exercises 1–6.

1. 7 to 14 **2.** 3 : 9 **3.** 5 : 50

4. $\frac{3}{18}$ **5.** 16 : 18 **6.** 24 to 15

7. There are 12 socks in Ryan's drawer. There are 9 black socks, and the rest are white. What is the ratio of black to white socks? Give your answer in its simplest form.

8. Chad works in a grocery store. He sells 22 apples in the same time that he sells 8 oranges. What is the ratio of apples to oranges sold? Give your answer in its simplest form.

You Can Simplify Ratios in Stages

Don't forget:
A common factor is a number that divides evenly into two (or more) other numbers. For example, 12 is a common factor of 24 and 36.

You can find the simplest form of a ratio in different ways. One way to do this is to make the ratio simpler in stages.

Example 3

Find the simplest form of the ratio 105 : 70.

Solution

Just keep looking for numbers that divide into both sides of the ratio (common factors).

105 and 70 both divide evenly by 5, so **105 : 70 is equivalent to 21 : 14**.
21 and 14 both divide evenly by 7, so **21 : 14 is equivalent to 3 : 2**.
Nothing divides into both 3 and 2, so 3 : 2 is in its simplest form.

So the **simplest form of 105 : 70 is 3 : 2**.

Don't forget:
You must remember to do the same thing to both sides of the ratio.

Another way is to find the **greatest common divisor** of the numbers in the ratio.

Example 4

Find the simplest form of the ratio 105 : 70.

Solution

The greatest common divisor of 105 and 70 is 35.
So divide both sides of the ratio by 35.

This means that the **simplest form of 105 : 70 is 3 : 2**.

Don't forget:
You can find the greatest common divisor (GCD) of two numbers by finding their prime factorizations.
$105 = 3 \times 5 \times 7$
$70 = 2 \times 5 \times 7$
Then multiply the common factors (5 and 7) together to find the greatest common divisor. So the GCD of 105 and 70 is $5 \times 7 = 35$.

Guided Practice

Find the simplest form of the ratios in Exercises 9–14.

9. 18 to 12 **10.** 6 : 9 **11.** 25 : 75

12. $\frac{24}{96}$ **13.** 72 : 56 **14.** 70 to 98

Ratios Can Contain Variables

Ratios can include **variables** as well as numbers.

Example 5

Find a number, x, so that the ratio 7 : 2 is equivalent to x : 4

Solution

You need to compare the ratios 7 : 2 and x : 4.
If you write the ratios as fractions, it's easier to compare them directly.

The two ratios are equivalent, so the fractions must be equal.

$$\frac{7}{2} = \frac{x}{4}$$

***x* stands for any number. You have to find what *x* is.**

The denominator of the second ratio is twice as big as the denominator of the first ratio. Since the fractions are equal, the numerator of the second ratio must also be twice as big as the numerator of the first.

So $x = 2 \times 7 = 14$.
This means **7 : 2 is equivalent to 14 : 4**.

Guided Practice

15. Find the number x, so that the ratio 9 : 3 is equivalent to 3 : x.

16. Find the number y, so that the ratio 14 : 44 is equivalent to y : 66.

Independent Practice

1. Mr. Jefferson has 7 boys in his science class, and 19 girls. What is the ratio of boys to girls?

2. There are 10 sculptures and 16 paintings in a room at the art museum. What is the ratio of sculptures to paintings? Give your answer in its simplest form.

In Exercises 3–6, write each ratio in its simplest form.

3. 5 to 10 **4.** 80 : 60 **5.** $\frac{18}{20}$ **6.** 48 to 33

7. Find a number y, so that the ratio 1 : 9 is equivalent to y : 18.

8. Find a number z, so that the ratio 5 : 6 is equivalent to z : 30.

For Exercises 9–10, give your answers in the simplest form.

A bag contains 40 beads, 25 coins, and 10 keys.
9. What is the ratio of beads to coins?
10. What is the ratio of keys to other objects in the bag?

Now try these:
Lesson 4.1.2 additional questions — p446

Round Up

So now you know how to find equivalent ratios, and how to simplify ratios. The next step is to write proportions, which is what you'll learn all about in the next Lesson.

Lesson 4.1.3

Proportions

Equivalent ratios are ones that can be simplified to be the same thing. In this Lesson, you'll learn that if two ratios are equivalent ratios, then you can write them as a proportion.

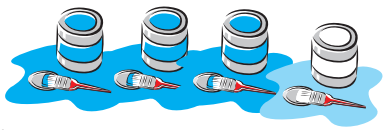
A Proportion Is an Equation of Equivalent Ratios

A **proportion** is an equation showing two equivalent ratios.
For example, the ratios 1 : 2 and 3 : 6 are equivalent.

So you can write the proportion: $\frac{1}{2} = \frac{3}{6}$

Example 1

To paint one wall, Gabrielle uses three cans of blue paint, and one can of white paint. To cover the room's four walls, she uses twelve cans of blue paint and four cans of white paint.



Write a proportion arising from this situation.

Solution

To write a proportion, you need two equivalent ratios.

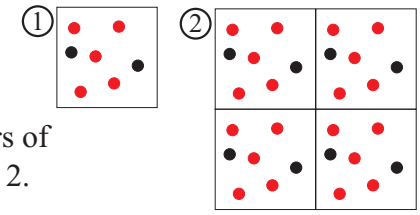
The first ratio comes from the paint Gabrielle used to paint **one wall**.
She used 1 can of white and 3 cans of blue.
This gives a ratio of white to blue of 1 : 3, or $\frac{1}{3}$.

The other ratio comes from the paint used to paint the **entire room**.
The ratio of white to blue for four walls is 4 : 12, or $\frac{4}{12}$.

You can write these two equivalent ratios as a proportion: $\frac{1}{3} = \frac{4}{12}$

Example 2

Pattern 2 consists of four tiles like Pattern 1.



Write a proportion involving the numbers of black dots and red dots in Patterns 1 and 2.

Solution

Pattern 1 contains **2 black dots** and **5 red dots**. This gives a ratio of $\frac{2}{5}$.

Pattern 2 must contain $4 \times 2 = $ **8 black dots** and $4 \times 5 = $ **20 red dots**.

This gives a ratio of $\frac{8}{20}$.

These ratios must be equivalent, so you can write the proportion $\frac{2}{5} = \frac{8}{20}$.

California Standard:

Number Sense 1.3

Use **proportions to solve problems (e.g., determine the value of N if 4/7 = N/ 21**, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

What it means for you:

You'll learn what proportions are, how to write them in math, and how to solve them.

Key words:

- proportion
- equivalent ratios

Check it out:

You can check whether your proportion is correct by simplifying both fractions until they are in their simplest form, and checking that they are equal.

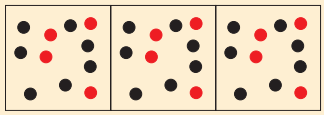
Guided Practice

1. The ratio of boys to girls in a class is 5 : 6.
The ratio of boys to girls in the whole school is 100 : 120.
Can these two ratios be written as a proportion? Explain your answer.

2. Can the ratios 3 : 7 and 5 : 21 be written as a proportion? Explain your answer.



3. The second pattern below is made up of three tiles like the first pattern.
Write a proportion using the numbers of black dots and red dots in the two patterns.



Don't forget:
Remember that you can write a proportion for any two ratios that are equivalent. You can't write proportions if the ratios aren't equivalent.

Proportions Can Involve Variables

In the last Lesson, you saw that ratios can involve variables.
The same is true of proportions.

Example 3

The ratios 3 : 4 and x : 12 are equivalent.
Write a proportion involving x.

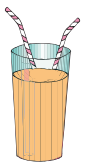
Solution
Just treat x like you would any other number.

So you can write the proportion $\frac{3}{4} = \frac{x}{12}$.

You may have to make a proportion describing a **word problem**.

Example 4

Meredith likes to mix her apple juice with a little water.
She uses a ratio of 1 part water to 4 parts apple juice.
If x is the number of cups of apple juice she would need to mix with 2 cups of water, write a proportion involving x.



Solution
You need two ratios to write a proportion.
Since Meredith needs to mix 2 cups of water with x cups of apple juice, one ratio of water to apple juice you can use is **2 to x**.

You also know Meredith wants a ratio of water to apple juice of **1 : 4**.

Use these two ratios to write a proportion: $\frac{2}{x} = \frac{1}{4}$

Don't forget:
The ratios "1 : 4," "1 to 4," and "$\frac{1}{4}$" all mean the same thing.

Guided Practice

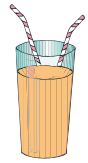
To make smoothies, Joshua uses a ratio of oranges to strawberries of 1 : 6. One morning, Joshua makes a smoothie using 3 oranges.
4. Call the number of strawberries Joshua uses x.
Write a proportion involving x.

Check it out:
This is exactly the same as Example 4 on the previous page, but this time you have to find x — not just write a proportion.

If you have a proportion containing a variable, you can **solve** it.

Example 5

Meredith likes to mix her apple juice with a little water. She uses a ratio of 1 part water to 4 parts apple juice. If x is the number of cups of apple juice she would need to mix with 2 cups of water, find x.



Solution

There are different ways to solve this...

(i) You can reason it out.
If the ratio of water to apple juice is 1 : 4, then there will be 4 times as much apple juice as water.

Since there are 2 cups of water, there will be **8 cups of apple juice**.

(ii) Or you can use your proportion: $\frac{2}{x} = \frac{1}{4}$.

The **numerator** on the right is half the numerator on the left. So the **denominator** on the right must be half the denominator on the left. This means $x = 4 \times 2 = 8$, and Meredith needs **8 cups of apple juice**.

Guided Practice

5. Write and solve a proportion for the equivalent ratios $x : 6$ and $4 : 12$.

To make smoothies, Joshua uses a ratio of oranges to strawberries of 1 : 6. One morning, Joshua makes a smoothie using 3 oranges.
6. Solve your proportion from Exercise 4 to find x.

Independent Practice

Now try these:
Lesson 4.1.3 additional questions — p446

In Exercises 1–6, decide whether the ratios are equivalent. For those ratios that are equivalent, write a proportion.

1. $\frac{2}{3}$ and $\frac{4}{6}$ **2.** $\frac{3}{5}$ and $\frac{5}{3}$ **3.** $\frac{4}{21}$ and $2 : 9$

4. $5 : 9$ and $25 : 45$ **5.** $4 : 2$ and $2 : 1$ **6.** $9 : 12$ and $81 : 108$

A recipe for carrot cake needs a ratio of carrots to walnuts of 2 : 3.
7. If 6 cups of walnuts are used with c cups of carrots, write a proportion involving c.
8. Solve your proportion to find c.

Round Up

Proportions can be useful in everyday life — you'll come across them countless times when you're cooking and painting, and so on. Next Lesson, you'll see another way to solve them to find unknowns.

Lesson 4.1.4

Proportions and Cross-Multiplication

California Standard:

Number Sense 1.3

Use proportions to solve problems (e.g., determine the value of N if 4/7 = N/21, find the length of a side of a polygon similar to a known polygon). **Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.**

What it means for you:

You'll learn about the cross-multiplication method for solving problems involving proportions with a missing variable.

Key words:

- proportion
- cross-multiplication
- multiplicative inverse
- reciprocal
- ratio

In this Lesson, you'll learn a good way to solve proportions — the method of cross-multiplication.

Most of the ideas in this Lesson apply to fraction equations too, so it's worth concentrating extra hard during this Lesson.

Multiply the Diagonals in Cross-Multiplication

Take this proportion: $\frac{2}{3} = \frac{6}{9}$.

If you multiply the **numerator** of each fraction by the **denominator** of the other, you get two **products**. (This is called **cross-multiplication**.)

$\frac{2}{3} \times \frac{6}{9}$ → $3 \times 6 = 18$

$\frac{2}{3} \times \frac{6}{9}$ → $2 \times 9 = 18$

For **any** proportion, these two products are always **equal**.

Example 1

Use cross-multiplication to check that the ratios $\frac{2}{5}$ and $\frac{8}{20}$ are equivalent.

Solution

If the two ratios are equivalent, then you can write them as a proportion. Then you can check that the products of the two diagonals are equal.

Cross-multiplication gives: $\frac{2}{5} \times \frac{8}{20}$ → $5 \times 8 = \mathbf{40}$

$\frac{2}{5} \times \frac{8}{20}$ → $2 \times 20 = \mathbf{40}$

Both products are **equal**, so the **ratios are equivalent**.

Example 2

The ratios 1 : 9 and 4 : a are equivalent.
Use cross-multiplication to find the value of a.

Solution

The two ratios are equivalent, so write them as a proportion: $\frac{1}{9} = \frac{4}{a}$

Now cross-multiply, and remember that the products will be **equal**.

$\frac{1}{9} \times \frac{4}{a}$ → $9 \times 4 = \mathbf{36}$

$\frac{1}{9} \times \frac{4}{a}$ → $1 \times a = \boldsymbol{a}$

So $\boldsymbol{a = 36}$.

Don't forget:
If cross-multiplication gives you two different numbers, then the ratios are not equivalent.

Guided Practice

In Exercises 1–6, use cross-multiplication to decide whether the ratios are equivalent.

1. $\frac{4}{5}$ and $\frac{20}{25}$ **2.** $\frac{2}{3}$ and $\frac{8}{14}$ **3.** 9 : 12 and 12 : 16

4. 8 to 9, and 9 to 8 **5.** 15 : 18 and 20 : 24 **6.** $\frac{4}{3}$ and $\frac{7}{5}$

In Exercises 7–9, use cross-multiplication to find the variable in the proportions.

7. $\frac{z}{6} = \frac{7}{1}$ **8.** $\frac{3}{c} = \frac{1}{8}$ **9.** $\frac{1}{9} = \frac{9}{g}$

Use Cross-Multiplication to Solve Any Proportion

Sometimes cross-multiplication is just the **first step** on the way to finding the value of an unknown.

Example 3

Solve the proportion $\frac{2}{y} = \frac{3}{12}$ for y.

Solution

Cross-multiply to find: $2 \times 12 = 3 \times y$.
You can write this more simply as $3y = 24$.

This is an equation like you saw in Lesson 2.2.4.
So divide both sides by 3 to find y.

$$3y = 24$$
$$3y \div 3 = 24 \div 3$$
$$y = 8$$

Example 4

Solve the proportion $\frac{5}{16} = \frac{b}{8}$ for b.

Solution

Cross-multiply to find: $5 \times 8 = 16 \times b$, or more simply, $16b = 40$.
Now divide both sides by 16 to find b.

$$16b = 40$$
$$b = 2.5$$

Guided Practice

Find the missing variable in Exercises 10–13.

10. $\frac{3}{y} = \frac{9}{18}$ **11.** $\frac{4}{24} = \frac{1}{x}$ **12.** $\frac{8}{15} = \frac{c}{90}$ **13.** $\frac{16}{8} = \frac{h}{2}$

You Can Solve Real-Life Problems Using This Method

Example 5

The ratio of the mass of sand to cement in a particular type of concrete is 4.8 : 2. If 6 kg of sand are used, how much cement is needed?

Solution

You need to write a proportion, so you need two ratios of sand : cement.

The first ratio is given in the question, **4.8 : 2**.

The second ratio will involve a variable, c, which represents the mass of cement needed. So the other ratio is **6 : c**.

Write a proportion: $\frac{4.8}{2} = \frac{6}{c}$

Cross-multiply: $4.8 \times c = 2 \times 6$, or more simply, **$4.8c = 12$**.

Solve your equation for c: $c = 12 \div 4.8$, which gives **$c = 2.5$**.

So **2.5 kg of cement are needed**.

You can see why cross-multiplication actually works by using **multiplicative inverses**.

Example 6

Show that if $\frac{3}{4} = \frac{a}{b}$, then $3b = 4a$.

Show every step of your work. Do not use cross-multiplication.

Solution

This involves a little bit of algebra...

Remember that $\frac{3}{4} = 3 \times \frac{1}{4}$, and $\frac{a}{b} = a \times \frac{1}{b}$. (See Lesson 3.2.1.)

So the original equation is the same as $3 \times \frac{1}{4} = a \times \frac{1}{b}$.

- Multiply both sides of this equation by 4 (the **multiplicative inverse** of $\frac{1}{4}$) and b (the **multiplicative inverse** of $\frac{1}{b}$).

 This gives: $4 \times b \times 3 \times \frac{1}{4} = 4 \times b \times a \times \frac{1}{b}$.

- You can rearrange this as: $4 \times \frac{1}{4} \times b \times 3 = b \times \frac{1}{b} \times 4 \times a$

- But $4 \times \frac{1}{4} = 1$ and $b \times \frac{1}{b} = 1$, so this simplifies to **$b \times 3 = 4 \times a$**.

And that's it... you've shown that **$4a = 3b$**.

Don't forget:

There's more information in Lesson 3.3.1 about multiplicative inverses (also called reciprocals).
The important fact is that when you multiply a number by its multiplicative inverse, you get 1.

So $4 \times \frac{1}{4} = 1$ and $b \times \frac{1}{b} = 1$.

Don't forget:

Rearranging the order of things that are multiplied together uses the associative and commutative properties of multiplication. See Lesson 2.3.2 for more information

Guided Practice

14. Lois is mixing up purple paint in the ratio 5 parts blue to 3 parts pink. If Lois uses 15 cans of pink paint, how many cans of blue paint will she need to mix the correct purple?

15. A new school is due to open. In total, it will need to employ 2 staff members for every 28 pupils. The school is predicted to have 2744 pupils. How many staff members will it need to employ?

Independent Practice

In Exercises 1–4, say which ratios are equivalent by using cross-multiplication.

1. 5 : 8 and 20 : 32

2. $\frac{3}{4}$ and $\frac{48}{60}$

3. $\frac{7}{5}$ and $\frac{49}{35}$

4. 8 to 3, and 6 to 2

In Exercises 5–12, solve the proportions using cross-multiplication.

5. $\frac{6}{1}=\frac{x}{2}$ **6.** $\frac{4}{y}=\frac{16}{4}$ **7.** $\frac{x}{7}=\frac{6}{21}$ **8.** $\frac{2}{x}=\frac{9}{36}$

9. $\frac{9}{117}=\frac{3}{x}$ **10.** $\frac{5}{35}=\frac{10}{x}$ **11.** $\frac{2}{y}=\frac{4}{76}$ **12.** $\frac{y}{6}=\frac{1}{3}$

13. Roberta is going on vacation. She needs to take 3 skirts for each week's vacation. Is she right in taking 8 skirts for a 3-week vacation?

14. Eva notices that she is taught by 3 male teachers and 4 female. If this ratio is true for the whole school, how many male teachers would there be if there are 52 female teachers altogether?

Exercises 15–17 apply to Jose, who is making a fruit salad for his friends, using 7 oranges, 14 pears, 14 apples, 98 grapes, and 28 cherries. An average portion of fruit salad contains 7 grapes.

15. How many pears are in an average portion?

16. How many oranges are in an average portion?

17. How many cherries are in an average portion?

18. If $\frac{a}{b}=\frac{c}{d}$, then what can you say about ad and bc? Show every step of your work. Do not use cross-multiplication.

Now try these:
Lesson 4.1.4 additional questions — p446

Round Up

In this Lesson, you've learned how to solve proportion equations using cross-multiplication. The next Section uses this to solve proportion problems in geometry.

Lesson 4.2.1

Section 4.2 Similarity

This Lesson is about similarity in geometry. If two figures are similar, they are the same shape, though they don't have to be the same size. You're going to use proportions to find out more about similar shapes.

California Standard:

Number Sense 1.3

Use proportions to solve problems (e.g., determine the value of N if 4/7 = N/21, **find the length of a side of a polygon similar to a known polygon**).

Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

What it means for you:

You'll learn about similar shapes and be able to express relationships in geometry using ratios.

Key words:

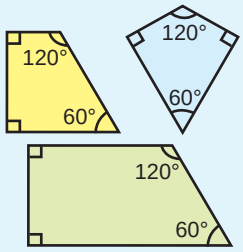
- corresponding sides/angles
- similar
- proportion
- ratio

Check it out:

If two figures are the same shape, and the same size, then they are called congruent.

Check it out:

Any two triangles whose angles have the same measures are similar. However, this doesn't work for other shapes — all three shapes below have the same angles, for example.

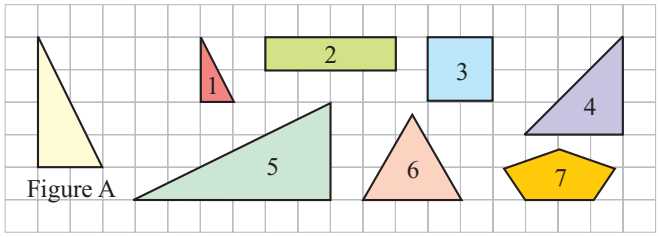


Similar Figures Are the Same Shape

Figures that have **exactly the same shape** (but could be **different sizes**) are called **similar** figures.

Example 1

Which of the shapes below are similar to Figure A?



Solution

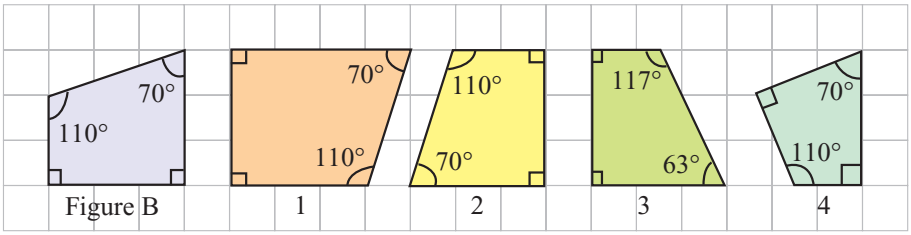
Shapes 2, 3, and 7 **don't have 3 sides**, so they're not similar to Figure A.
Shape 6 **doesn't have a right angle**, so that's not similar to Figure A.
Shape 4 **has two sides the same length**, so that's not similar to Figure A.

Shape 1 is smaller than Figure A, but is the **same shape**.
Shape 5 is bigger than Figure A and has been rotated, but it is also the **same shape**. **Shapes 1 and 5 are similar to Figure A.**

To say whether two shapes are similar, you need to look at their **angles** and **side lengths**.

Example 2

Which one of the shapes on the right is similar to Figure B below?



Solution

Shape 3 has different angles from Figure B, so that isn't similar.
Shape 1 has the same angles, but it's been stretched in only one direction, making it a different shape, and not similar to Figure B.
Shape 4 has the same angles, but the two right angles are opposite each other — different from Figure B. So Figure B and Shape 4 are not similar.

Shape 2 has the same angles and the same side lengths. It could be cut out and placed exactly over Figure B. **Shape 2 is similar to Figure B.**

Guided Practice

1. Which one of the shapes 1–4 below is similar to Figure C? Explain your answer.

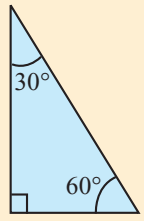
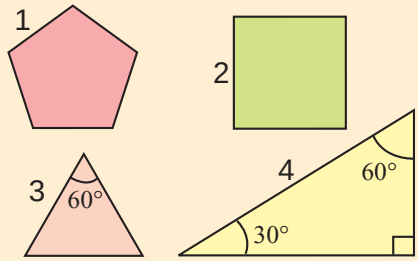


Figure C

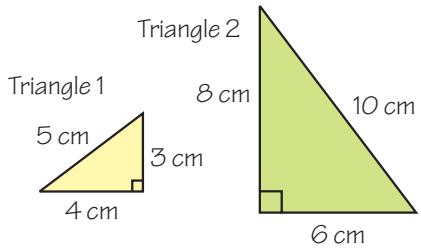


Check it out:
Basically, similar shapes are enlargements or reductions of each other (but can be reflected and rotated too).

Sides of Similar Shapes Give the Same Ratios

The two triangles below are similar.

Look at the **ratios** of the lengths of corresponding sides.



Shortest sides: $\frac{3\text{ cm}}{6\text{ cm}} = \frac{1}{2}$,

medium sides: $\frac{4\text{ cm}}{8\text{ cm}} = \frac{1}{2}$, and longest sides: $\frac{5\text{ cm}}{10\text{ cm}} = \frac{1}{2}$.

Check it out:
This ratio of $\frac{1}{2}$ means that the lengths of the sides in Triangle 1 are $\frac{1}{2}$ as big as the corresponding sides in Triangle 2.

The ratios of the lengths of corresponding sides are **equal**. This is true for all **similar** shapes.

Example 3

All these triangles are similar. Find the ratios of the lengths of corresponding sides between Figure D and:

(i) Shape 1
(ii) Shape 2
(iii) Shape 3

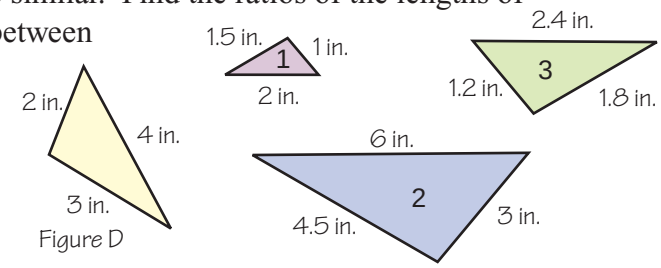


Figure D

Solution

Since all the shapes are similar, the ratios of all corresponding side lengths will be equal.

(i) For Shape 1: $\frac{2\text{ in.}}{1\text{ in.}} = \frac{3\text{ in.}}{1.5\text{ in.}} = \frac{4\text{ in.}}{2\text{ in.}} = 2$

(ii) For Shape 2: $\frac{2\text{ in.}}{3\text{ in.}} = \frac{3\text{ in.}}{4.5\text{ in.}} = \frac{4\text{ in.}}{6\text{ in.}} = \frac{2}{3}$

(iii) For Shape 3: $\frac{2\text{ in.}}{1.2\text{ in.}} = \frac{3\text{ in.}}{1.8\text{ in.}} = \frac{4\text{ in.}}{2.4\text{ in.}} = \frac{5}{3}$

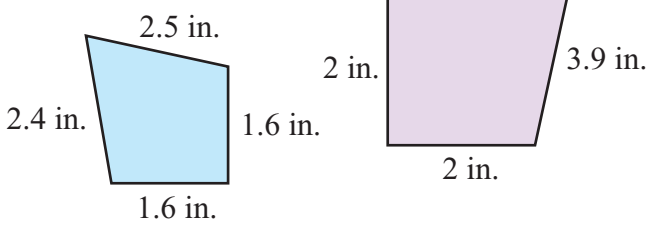
Don't forget:
An equation showing that two ratios are equal is a proportion.
Here, you have lots of ratios equal to each other, so there are lots of proportions.
For example, $\frac{4\text{ in.}}{2.4\text{ in.}} = \frac{5}{3}$ is a proportion.

Check it out:

This time, you can't use the fact that similar shapes have identical angles. You have to use what you know about corresponding sides of similar shapes — that they all give the same ratio.

Example 4

Are the two shapes below similar?
Explain your answer.



Solution

Work out the ratios of the lengths of corresponding sides.
If the shapes are **similar**, all the ratios will be **equal**.

Shortest sides: $\frac{1.6 \text{ in.}}{2 \text{ in.}} = \mathbf{0.8}$

Longest sides: $\frac{2.5 \text{ in.}}{4 \text{ in.}} = \mathbf{0.625}$

You don't have to work out the ratio for the other sides — these two ratios are different, so you know the shapes aren't similar.

The ratios of the lengths of the sides are **not all equal**, so **the shapes are not similar**.

Guided Practice

All these triangles are similar. Find the ratios of the lengths of corresponding sides between Figure E and:

2. Shape 1

3. Shape 2

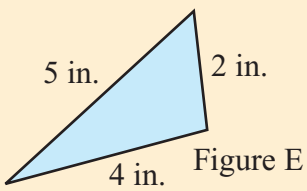
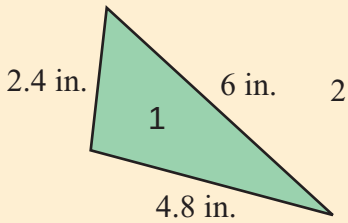
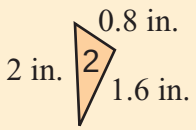
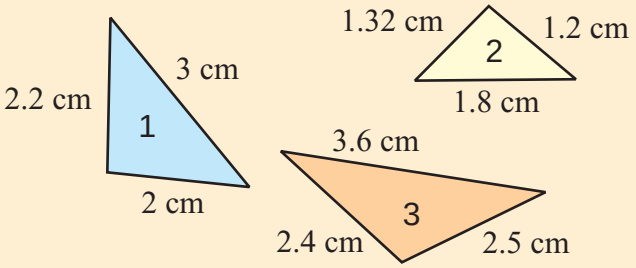


Figure E





4. Which two shapes below are similar?
Explain your answer.



Don't forget:
If you find it easier to think about pictures, draw some. A quick sketch of two triangles with the lengths of the sides marked in can be much easier to think about than a description.

Example 5

Two triangles are similar.
The first triangle has sides of length 7 in., 8 in., and 9 in.
The second triangle has sides of length 14 in. and 16 in., but the length of the longest side is unknown.

Calculate the ratio between the lengths of corresponding sides.
Use this ratio to find the length of the missing side.

Solution

Work out the side-length ratios of the bigger triangle to the smaller triangle.

Using the shortest sides: $\frac{14 \text{ in.}}{7 \text{ in.}} = 2$

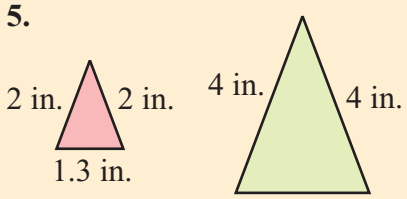
So sides in the bigger triangle are **twice as long** as in the smaller triangle.

So the length of the longest side of the bigger triangle must be 2 × 9 in. = **18 inches**.

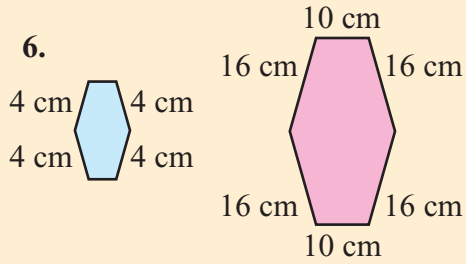
Guided Practice

In Exercises 5–6, the shapes in each pair are similar.
Find the missing lengths by working out the ratios of the corresponding side lengths.

5.

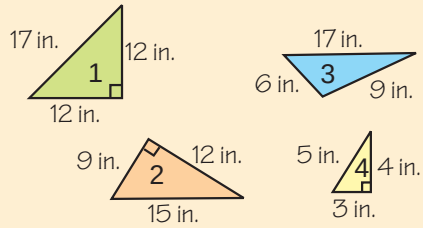


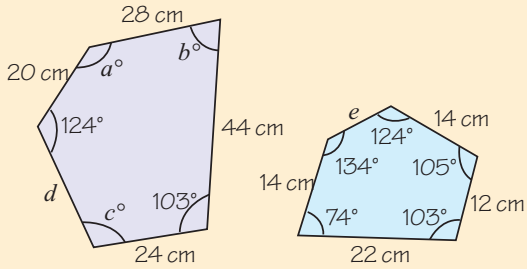
6.



Independent Practice

1. Show that triangles 2 and 4 are similar to each other, but that triangles 1 and 3 are not similar to any others.





2. Fill in the missing lengths and angles in the similar pentagons on the left.

Now try these:
Lesson 4.2.1 additional questions — p447

Round Up

Now you know how to recognize similar figures, and work out missing angles and lengths.
Next Lesson, you'll use proportions to solve more problems with similar shapes.

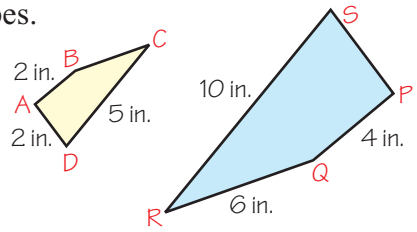
Lesson 4.2.2

Proportions and Similarity

Last Lesson, you learned about similar shapes. You saw that if you take two similar triangles and work out the ratio of corresponding sides, you always get the same answer. This Lesson, you're going to use proportions to find missing lengths. But first, a bit about notation...

You Can Use Proportions to Find Missing Lengths

Look at these **similar** blue and yellow shapes.
To name a particular **side**, you write the letters at either end with a **bar** over them. So the longest side of the yellow shape is $\overline{CD}$, while the corresponding side of the blue shape is $\overline{RS}$.



The **lengths** of sides are written in the same way, but **without** the bar. So the length of $\overline{CD}$ is written **CD**.

You've seen that for **similar** shapes, the ratio of the lengths of **any** two corresponding sides is equal. You can use this to write a **proportion**.

Example 1

Look at the blue and yellow shapes above.
Write a proportion using the ratios AB : PQ and BC : QR.

Solve your proportion to find the missing length BC.

Solution

These are similar shapes, so the ratios of the lengths are the same.

This means $\frac{AB}{PQ} = \frac{BC}{QR}$.

If the length of the missing side is z **inches**, then $\frac{2 \text{ in.}}{4 \text{ in.}} = \frac{z \text{ in.}}{6 \text{ in.}}$,
or more simply, $\frac{2}{4} = \frac{z}{6}$.

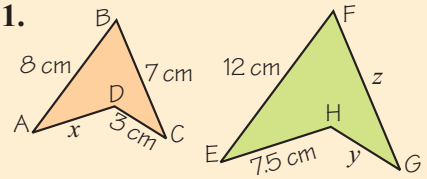
Cross-multiplication then gives $4z = 6 \times 2$, or $4z = 12$.
Now you can divide both sides by 4 to get $z = 12 \div 3 = 4$.

The length of side BC is 4 inches.

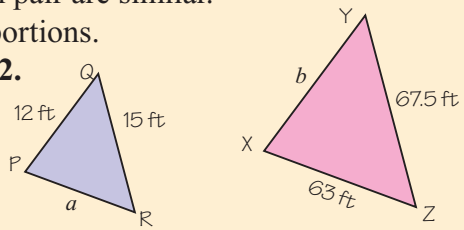
Guided Practice

In Exercises 1–2, the shapes in each pair are similar.
Find the missing lengths using proportions.

1.



2.



California Standard:

Number Sense 1.3
Use proportions to solve problems (e.g., determine the value of N if 4/7 = N/21, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

What it means for you:
You'll use proportions to find missing lengths in similar shapes.

Key words:
- proportion
- similar shapes
- ratio
- corresponding sides

Don't forget:
These ratios can be written in different ways.
(i) AB : PQ
(ii) AB to PQ
(iii) $\frac{AB}{PQ}$

Don't forget:
The ratio of the length of corresponding sides in similar triangles will always be the same. But you must make sure you get the numbers in the right order. In Example 1, the yellow triangle's sides are always on the top.

Check it out:
This is the same method as was used in Example 5 of the previous Lesson, only written down using proportions.

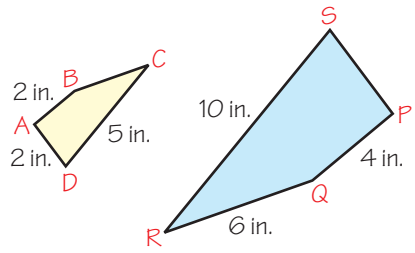
There are other ratios you can use to find missing lengths in similar shapes.

Example 2

Look at the shapes on the right.

Calculate the ratios $\frac{AB}{CD}$ and $\frac{PQ}{RS}$.

What do you notice?



Solution

Here, you're working out ratios of **two sides on the same shape**.

First look at the yellow shape: $\frac{AB}{CD} = \frac{2 \text{ in.}}{5 \text{ in.}} = \frac{2}{5}$

Now look at the blue shape: $\frac{PQ}{RS} = \frac{4 \text{ in.}}{10 \text{ in.}} = \frac{4}{10} = \frac{2}{5}$

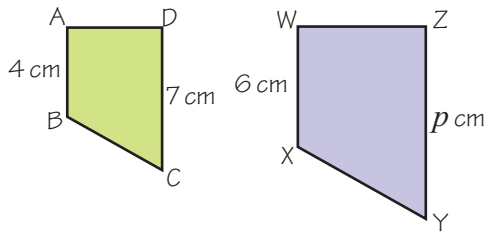
The two ratios are equal.

This is always true for similar shapes. If you work out the **ratio of the lengths of two sides in one shape**, then the ratio of **the corresponding pair of sides** in any similar shape will be the same.

Example 3

Look at these similar shapes. Write a proportion using the ratios AB : CD and WX : YZ.

Solve your proportion to find p.



Solution

The ratio AB : CD is $\frac{4 \text{ cm}}{7 \text{ cm}} = \frac{4}{7}$. This is the **ratio of the lengths of the parallel sides** on the green trapezoid.

Since the purple shape is **similar**, the ratio of the lengths of its parallel sides will also equal $\frac{4}{7}$. So $\frac{6}{p} = \frac{4}{7}$.

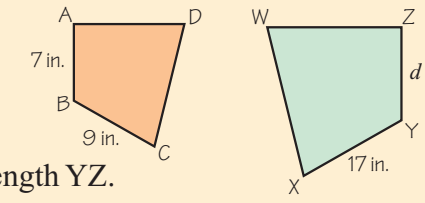
Now solve your proportion using cross-multiplication:

$4p = 42$, which gives $p = 10.5$.

Check it out:
Similar shapes are sometimes described as being "in proportion."

Guided Practice

For Exercises 3–6, use the similar shapes in the diagram below.



3. What is the ratio AB : BC?

4. What is the ratio YZ : XY?

5. Use your answers from Exercises 3 and 4 to set up a proportion.

6. Solve your proportion to find the length YZ.

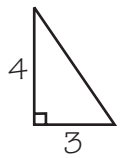
In questions about lengths, make sure you remember about **units**.

Example 4

The two shortest sides of a right triangle are in the ratio 4 : 3.
How long is the shortest side of a similar triangle, if its second shortest side is 17.5 miles long?

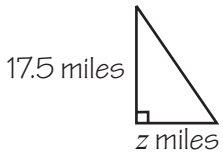
Solution

Here, the question doesn't give you a picture.
In cases like this, it's a good idea to draw one yourself.



Here's a right triangle whose two shortest sides are in the ratio 4 : 3.

You know that the second shortest side of a similar triangle is 17.5 miles long.
Call the length you need to find z miles.



Now you can set up a proportion.
You have a choice which ratios to use, but the math is the same.

(i) You can either use corresponding sides: $\frac{17.5}{4} = \frac{z}{3}$
Cross-multiply: $3 \times 17.5 = 4z$
Divide both sides by 4 to get: $z = 52.5 \div 4 =$ **13.125**

(ii) Or you can use the ratio of the two short sides in each triangle: $\frac{17.5}{z} = \frac{4}{3}$
Cross-multiply: $3 \times 17.5 = 4z$
Divide both sides by 4 as before to get: $z =$ **13.125**

Don't forget the units. Your final answer is that the shortest side of the triangle has length **13.125 miles**.

Guided Practice

An architect designs a building with a triangular wall.
The ratio of the lengths of the two shortest sides of the triangle is 2 : 5.
On the model of the building, the shortest side has length 8 cm.

7. What is the length of the second shortest side on the model?
8. On the actual building, the length of the shortest side of the triangular wall is 4 m. Write and solve a proportion to find the length of the second shortest side of the wall.

Don't forget:
If there's no picture in the question, you should probably draw one yourself.

Don't forget:

Use what you know about one triangle to help you find out about the other.

Don't forget:

If you're not sure where to start a question, write down some information about the triangles. Include a diagram, plus ratios of corresponding sides. Remember, if shapes are similar, there will be some equal ratios.

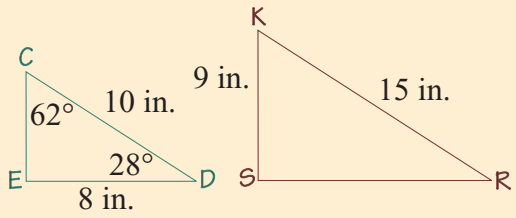
Now try these:

Lesson 4.2.2 additional questions — p447

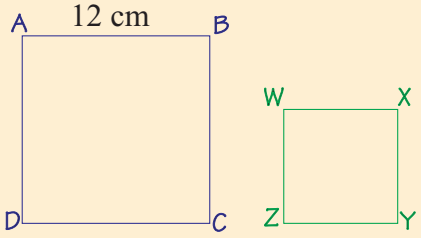
Independent Practice

Exercises 1–3 are about the two similar triangles below.

1. Find the angle at S.
2. Find the length SR.
3. Find the length CE.

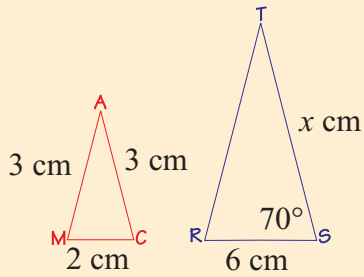


4. The ratio of the corresponding sides of these two squares is 4 : 3. Find the length XY.



These two triangles are similar.

5. Find the length of the missing side x.
6. Determine the measure of the angle at C.
7. What is the measure of the angle at M?



8. The ratio of corresponding sides of two similar shapes is 7 : 10. If the smaller shape has a side of length 21 in., find the length of the corresponding side in the larger shape.

9. One sheet of paper is 8.5 inches by 11 inches. Another sheet of paper is 16 inches by 22 inches. Are the two sheets of paper similar?

10. Alisha has a picture that is 12 cm wide and 18 cm long. She makes a poster that is similar to the original picture, but has a width of 30 cm. What is the length of the poster?

11. A quadrilateral has a perimeter of 7 inches. The ratio of the length of the sides in this quadrilateral and a similar one is 14 : 37. What is the perimeter of the second quadrilateral?

Round Up

That's it for similar shapes. Next, you'll be looking at scale drawings, which will use what you know about similar shapes and proportion and link it to real-life things such as maps.

Lesson 4.2.3

Scale Drawings

In this Lesson, you'll learn about scale drawings. Scale drawings are really just similar shapes. Any kind of object, from a huge airplane to a tiny mechanical component, can be drawn to scale. This is useful when an object is just too big or too small to draw life-size.

Scale Is a Kind of Ratio

Scale drawings are used when an object is **too large** or **too small** to draw full-size.

The scale is the **ratio** of any length in the drawing to the corresponding length in real life. So the scale tells you how much bigger or smaller the drawing is than the real object.

Sometimes the scale can be given using **units**.

Example 1

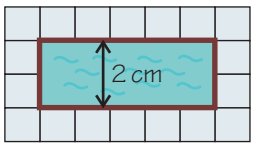
An architect made a scale drawing of a recreation center, using a scale of 1 cm : 3 meters. In the drawing, the swimming pool is 2 cm wide.

How wide is the actual pool?

Solution

The scale is 1 cm : 3 meters. So 1 cm on the drawing represents 3 meters in real life.

This means that if the pool is 2 cm wide on the drawing, the real pool will be **6 meters wide**.



Sometimes the ratio involves just **numbers**.

Example 2

The same architect has drawn a scale drawing of another recreation center. This time, she used a scale of 1 : 500.

The center's pool is 50 m long. How long is the pool in the drawing?

The pool is tiled. If the actual length of each tile is 15 inches, how long would each tile be drawn?

Solution

This time the scale is 1 : 500.
So **1 cm** on the drawing represents **500 cm** in real life, and so on.

If the actual pool is **50 meters** long, then the drawing will be:
(50 ÷ 500) meters = 0.1 m = 10 cm long.

The scale of 1 : 500 also means **1 inch** on the drawing represents **500 inches** in real life. So if a tile is 15 in. long, then in the drawing they would be: (15 ÷ 500) inches = **0.03 inches long**.

California Standards:

Number Sense 1.2

Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) **to show the relative sizes of two quantities**, using appropriate notations (a/b, a to b, a:b).

Number Sense 1.3

Use proportions to solve problems (e.g., determine the value of N if 4/7 = N/21, find the length of a side of a polygon similar to a known polygon). **Use cross-multiplication** as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

What it means for you:

You'll learn how to make a scale drawing, and use scales to find real lengths from a scale drawing, and vice versa.

Key words:

- scale
- ratio
- proportion
- map
- dimensions

Guided Practice

Lenny has a drawing of a boat, made with a scale of 1 inch : 15 feet.
1. The length of the boat in the drawing is 4 inches.
What is the length of the real boat?
2. The real boat is 30 feet wide. How wide is the boat in the drawing?
3. The real boat's mast is 20 feet high.
How high is the mast in the drawing?

Mr. Michael's house is drawn using a scale of 1 : 144.
In the drawing, a rectangular window is 0.25 in. by 0.5 in.
4. What are the dimensions of the actual window in inches?
5. How tall is an actual door if it's drawn 0.05 feet tall?

6. A computer contains a tiny component 0.1 mm long.
If a drawing is made of this component using a scale of 100 : 1, how long will the component be in the drawing?

Check it out:
In Exercise 6, the scale drawing is bigger than the actual component. Remember, a scale is written in the order "drawing : real-life."

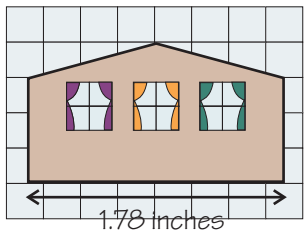
Just like with other ratios, you can use proportions to find missing lengths.

You have to be really careful with **units** though. One way to deal with units is to **multiply** and **divide** by them as though they are **numbers**.

Example 3

This is a scale drawing of the side of a house.
It is drawn using a scale of 2.5 inches : 24 feet.

Find the actual width of the side of the house.
(Each square on the paper is 0.25 inches wide.)



Solution
Call the actual width of the house w.

To make a proportion, you need two ratios.
You can always use the scale for one of these ratios: $\frac{\textbf{2.5 inches}}{\textbf{24 feet}}$.

For the other ratio, you need to use any measurement on the **drawing**, and the **corresponding measurement** in real life. Here, you're interested in the width of the house, so that's the measurement to use.

Ratio of **width in drawing to width in real life** is $\frac{\textbf{1.78 inches}}{\boldsymbol{w}}$.

Use your two ratios to write a proportion: $\frac{\textbf{1.78 inches}}{\boldsymbol{w}} = \frac{\textbf{2.5 inches}}{\textbf{24 feet}}$

Now cross-multiply, treating the units as though they are numbers.

$w \times 2.5 \text{ inches} = 1.78 \text{ inches} \times 24 \text{ feet}$
$w \times 2.5 = 1.78 \times 24 \text{ feet}$ — **Divide both sides by "inches"**
$2.5w = 42.72 \text{ feet}$ — **Multiply out 1.78 × 24**
$w = 42.72 \text{ feet} \div 2.5$ — **Divide both sides by 2.5**
$\boldsymbol{w = 17.088 \text{ feet}}$

Don't forget:
You solved proportions last Section. If you can't remember the cross-multiplication method, go back to 4.1.3 and recap.

Guided Practice

7. Find the actual height of the house from Example 3.

Miguel makes a drawing of an aircraft for a school project.
The scale of the drawing is 3 inches : 10 feet.
The length of the aircraft in Miguel's drawing is 14.7 inches.
8. Let the length of the real aircraft be z.
Write a proportion involving z.
9. Solve your proportion to find z.
10. If the actual aircraft is 44.5 feet wide, use a proportion to find how wide the aircraft will be in the drawing.

Sarah has made a scale drawing of a house using a scale of 1 : s.
The height of the house in the drawing is 4 inches.
The height of the actual house is 28 feet.
11. Find the number s.
12. If the actual height of the front door is 7 feet, use a proportion to calculate how tall the front door is in the drawing.

Don't forget:
To use a proportion to find a missing length, you need two equivalent ratios. Use the scale as one ratio.

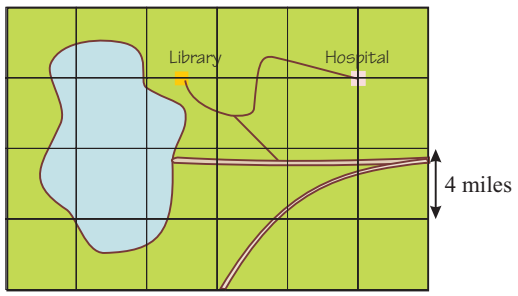
A Map Is a Scale Drawing of an Area

A **map** is another example of a **scale drawing**.

Example 4

Write and solve a proportion to find the distance between the library and the hospital.

The scale of the map is 1 grid square : 4 miles.



Solution

To write a proportion, you need two ratios.

From the scale, $\frac{\textbf{1 grid square}}{\textbf{4 miles}}$ — use this as one of your ratios.

Call the actual distance d. Then $\frac{\textbf{2.5 grid squares}}{\boldsymbol{d}}$ is an equivalent ratio.

Your proportion is then: $\frac{\textbf{1 grid square}}{\textbf{4 miles}} = \frac{\textbf{2.5 grid squares}}{\boldsymbol{d}}$

$d \times 1$ grid square $= 4$ miles $\times 2.5$ grid squares **Cross-multiply**
$d \times 1 = 4$ miles $\times 2.5$ **Divide both sides by the units "grid square"**
$d = 10$ miles **Simplify the equation**

The distance between the library and the hospital is 10 miles.

Guided Practice

Exercises 13–15 are about the same map. On this map, the distance between town A and town B is 4 inches. The actual distance between the towns is 8 miles.

13. What is the scale of the map in the form 1 in. : x miles?

14. If the real distance between towns C and D is 17 miles, how far apart would they be on the map?

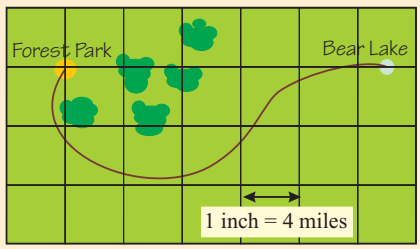
15. If the distance between towns E and F on the map is 0.5 inches, how far apart are they in real life?

16. The actual distance from Green City to Blue River is 160 miles. The distance between these cities on a map is 2.5 inches. If the distance on the same map between Blue River and Yellow Grove is 3 inches, what is the actual distance between these two places?

Independent Practice

1. Miguel makes a model of an aircraft for a school project. The scale of the model is 1 inch : 10 feet. If the length of Miguel's model is 20 inches, what is the length of the actual aircraft?

Jenna is taking a car trip from Forest Park to Bear Lake. The grid squares on the map measure 1 inch × 1 inch.



2. How many inches on the map is Forest Park from Bear Lake? Give the straight-line distance.

3. What is the straight-line distance of Bear Lake from Forest Park in real life?

4. The actual length of the road between the two places is 37 miles. On the map, how long is the line representing the road? Use a proportion to calculate your answer.

The length of a tiny organism is 0.4 mm. The length of a scale drawing of the same organism is 3.2 cm.

5. The scale of the drawing is p : 1. Find p.

6. If the actual organism has a width of 0.2 mm, what is the width of the organism on the scale drawing?

7. Harriet has two maps of the same city. The first map has a scale of 1 inch : 2 miles, and the second map has a scale of 1 inch : 3 miles. The distance from City Hall to the Lake on the first map is 2.5 inches. What is this distance on the second map?

Now try these:
Lesson 4.2.3 additional questions — p448

Round Up

This is all useful information. You'll almost certainly need to use these skills again — whenever you're reading a map, for instance. Next Lesson, you'll look at using ratios to convert between different units.

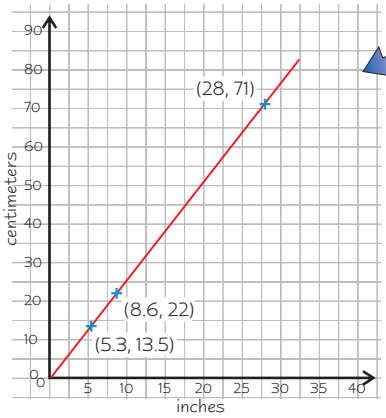
Section 4.3 introduction — an exploration into:
Plotting Conversions

How tall is your textbook? There are different answers. You could say that it's about 8.5 inches. Your friend might say that it's almost 22 centimeters. Someone else might say that it's about 0.00013 miles. That's what this Exploration is all about — measurements and units.

Choose **five** objects in the classroom that are **no more than 3 feet long**.

Measure the exact **length** of each object twice: (i) in **inches**, and (ii) in **centimeters**.
Record your measurements in a chart like this one.

Object	Length in inches	Length in centimeters
Textbook	8.6	22
Pen	5.3	13.5
Height of desk	28	71



Now draw a **coordinate grid** like this one. Make sure that your axes are **long enough** to show your measurements.

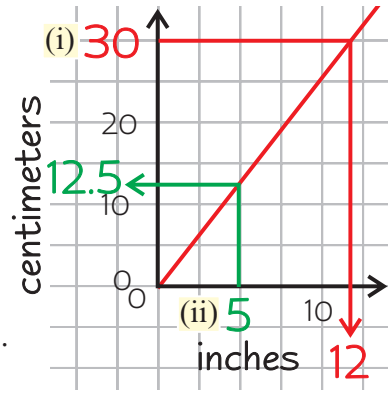
Plot each object's measurements as a point on the grid.
- Plot the number of **inches** along the ***x*-axis** (the horizontal axis).
- Plot the number of **centimeters** up the ***y*-axis** (the vertical axis).

You should be able to draw a **straight line** though the points.

Now you can use your graph to **convert** measurements in inches to centimeters, and measurements in centimeters to inches.

Example

Use your graph to convert: (i) 30 cm to inches,
(ii) 5 inches to centimeters.



Solution

(i) First find 30 on the **centimeters axis** (the vertical axis).
Go **across** to your plotted line, and **down** to the **inches axis**.
This tells you that 30 centimeters is about **12 inches**.

(ii) This time find 5 on the inches axis (the horizontal axis).
Go **up** to your plotted line, and **across** to the **centimeters axis**.
This tells you that 5 inches is about **12.5 centimeters**.

Exercises

1. Use your graph to convert these measurements from inches to centimeters, and from centimeters to inches.
a. 45 cm **b.** 34 inches **c.** 10 cm **d.** 10 inches

Round Up

You can do the same thing for other units too. For example, you could find the mass of a few different objects in both grams and ounces, plot the results on a graph, and then use your graph to convert other masses from one set of units to the other. You'll see lots about units in this Section.

Lesson 4.3.1

Section 4.3
Customary and Metric Units

This Lesson, you'll learn how to convert between units in either the customary or the metric system. So you take a length in inches, say, and convert it to feet. Or maybe you know a length in centimeters and you need to convert it to kilometers. That kind of thing...

California Standards:

Algebra and Functions 2.1
Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

Mathematical Reasoning 3.1
Evaluate the reasonableness of the solution in the context of the original situation.

What it means for you:
You'll see how to convert between different units within the customary or metric systems. For example, feet to inches, or meters to kilometers.

Key words:
- conversion
- customary system
- metric system
- unit
- conversion table
- conversion factor

Check it out:
Since 1 yard = 36 inches, the conversion factor between inches and yards is 36.

Check it out:
You should memorize the conversion factors in the table above.

Customary Lengths Are Inches, Feet, Yards, and Miles

The **customary system** of lengths is the system that uses **inches, feet, yards, and miles**.

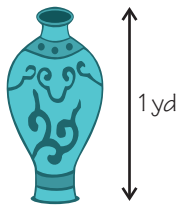
This **conversion table** shows how many of one unit make up another kind of unit.

Customary Lengths	
1 foot (ft)	= 12 inches (in.)
1 yard (yd)	= 3 feet (ft)
1 mile (mi)	= 1760 yards (yd)

You can use it to **convert** a measurement in one unit to a measurement in another unit. The numbers in a conversion table (12, 3, and 1760 in the above table) are sometimes called **conversion factors**.

Example 1

A vase is 1 yard tall. What is its height in:
(i) feet? (ii) inches?



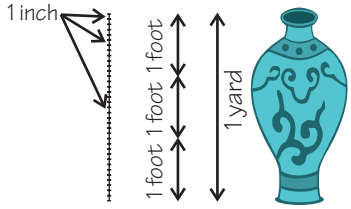
Solution

(i) Using the conversion table, you can see that 1 yard is **3 feet**.

(ii) If 1 foot is 12 inches, then 3 feet must be $3 \times 12 =$ **36 inches**.

Notice that when you convert to a **smaller unit**, the **number gets bigger**.

This makes sense... the smaller each unit, the more of them you need to cover the same distance.



In fact, if your conversion factor is greater than 1:
- you multiply by it to convert to a **smaller unit**, since this gives you a **bigger number**.
- you divide by it to convert to a **bigger unit**, since this gives you a **smaller number**.

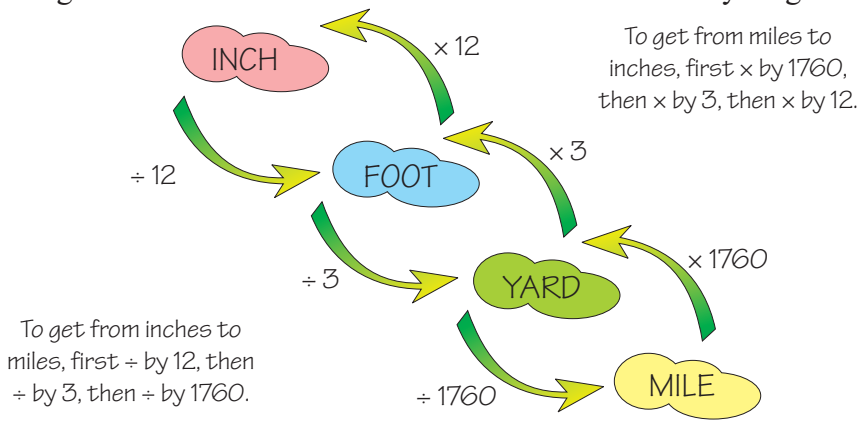
Example 2

Convert: (i) 3 miles to yards, (ii) 7920 yards to miles.

Solution

There are 1760 yards in a mile, so use 1760 as your conversion factor.
(i) Yards are smaller than miles, so 3 miles $= 3 \times 1760 =$ **5280 yd**
(ii) Miles are bigger than yards: 7920 yd $= 7920 \div 1760 =$ **4.5 miles**

This diagram summarizes how to convert between customary lengths.



Don't forget:

Always check that your answers are sensible. If you're converting 7 miles to yards, say, you'd expect your answer to be a very big number.
You could use estimation to check your answers (see Section 1.4 for more information).
For example, if you're converting 7 miles to yards, you'd expect an answer bigger than 7 × 1000 = 7000, but less than 7 × 2000 = 14,000.

Guided Practice

Convert:

1. 7 miles to yards
2. 8800 yards to miles
3. 72 inches to feet
4. 12 feet to yards
5. 5 feet to inches
6. 9 yards to inches
7. 1 mile to feet
8. 8 yards to inches
9. 2 miles to inches
10. 100,000 inches to miles

11. If Letitia cycles 3 miles, how many feet has she cycled?

12. Mount Whitney is 14,494 feet tall. How many miles is this?

Check it out:

Dekameters and hectometers aren't used very much at all. You also don't see decimeters very often.

Check it out:

In the metric system, you can work out how big a unit is by its prefix.
The prefix "kilo-" means thousand, so 1 kilometer is 1000 meters.
"deka-" means ten,
"hecto-" means hundred,
"kilo-" means thousand,
"deci-" means tenth,
"centi-" means hundredth,
"milli-" means thousandth.
This system works for other kinds of metric units too, not just for units of length.
For example, 1 kilogram is 1000 grams.
You should memorize these prefixes and what they mean — especially: "kilo-", "centi-", and "milli-".

Don't forget:

Dividing by 0.001 (= $\frac{1}{1000}$) is the same as multiplying by 1000.
See Lesson 3.3.1 for more information.

The Metric System Is Based on the Meter

The **metric system** for lengths is based on the **meter**. All metric units for length contain the word "meter" — millimeter, centimeter, decimeter...

Here's a conversion table.

Metric Lengths	
1 dekameter (dam) = 10 meters (m)	1 decimeter (dm) = 0.1 meters
1 hectometer (hm) = 100 meters	1 centimeter (cm) = 0.01 meters
1 kilometer (km) = 1000 meters	1 millimeter (mm) = 0.001 meters

Dekameters, hectometers, and kilometers are all bigger than meters.

Decimeters, centimeters, and millimeters are all smaller than meters.

Notice in this table that some of the conversion factors are less than one.

Example 3

Convert 120 meters to millimeters.

Solution

You're converting from a big unit (meters) to a smaller one (millimeters), so the number needs to be **bigger**.

The conversion factor from the table is 0.001. This is less than 1, so to make a **bigger number**, you need to **divide** by the conversion factor.
120 m = 120 ÷ 0.001 = 120 × 1000 = **120,000 mm**

Don't forget:
A bigger unit means a smaller number.

Example 4

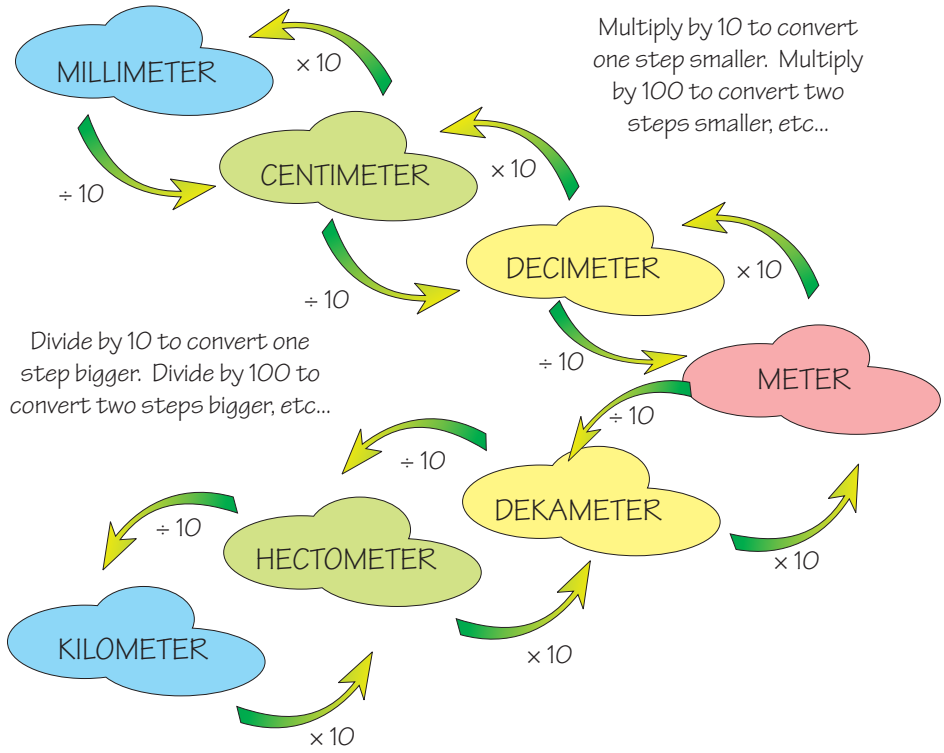
A pencil is 10 cm long. How long is the pencil in kilometers?

Solution

First, convert 10 cm to meters by multiplying by 0.01. This gives a **smaller number**: 10 centimeters = 10 × 0.01 = **0.1 m**

Next, convert 0.1 m to kilometers by dividing by 1000. This also gives a **smaller number**: 0.1 m = 0.1 ÷ 1000 = **0.0001 km**

This diagram summarizes how to convert between metric lengths.



Check it out:
Notice that:
(i) there are always two conversion factors.
This is because
1 cm = 0.01 m (giving a conversion factor of 0.01),
but 1 m = 100 cm (giving a conversion factor of 100), and
(ii) these conversion factors always multiply together to give 1 (for example, 0.01 × 100 =1).

This means that, to convert meters to centimeters, you can either:
(i) multiply by 100, or
(ii) divide by 0.01.

Guided Practice

Don't forget:
Check that all your answers are sensible.

Convert:

13. 7 kilometers to meters
14. 8700 meters to kilometers
15. 85 mm to cm
16. 120 cm to mm
17. 180 m to km
18. 15,000 m to km
19. 0.1 km to m
20. 160 mm to cm

21. A snail moves 3 m. How many cm is this?

Independent Practice

There are 12 inches in 1 foot, meaning the conversion factor is 12.
1. Would you divide by 12 or multiply by 12 when converting from feet to inches?
2. Would you divide by 12 or multiply by 12 when converting from inches to feet?

3. How would you convert from yards to miles?

In Exercises 4–9, find the missing values in the conversions.

4. 36 in. = ? yd	**5.** 144 in. = ? yd	**6.** 27 ft = ? yd
7. 100 yd = ? ft	**8.** 99 ft = ? yd	**9.** 0.25 mi = ? ft

10. TJ's guitar is 3.5 ft long.
What is the length of the guitar in inches?

11. Would you divide by 100 or multiply by 100 when converting from meters to centimeters?

12. There are 1000 millimeters in a meter, and 100 meters in a hectometer. What is the conversion factor when converting millimeters to hectometers?
13. Would you multiply or divide by this conversion factor when converting millimeters to hectometers?

Find the missing value in Exercises 14–19.

14. 30 m = ? cm	**15.** 2 km = ? m	**16.** 250 mm = ? cm
17. 0.5 cm = ? m	**18.** 4 dam = ? dm	**19.** 0.37 dm = ? mm

20. An Olympic swimming pool has a length of 50 meters.
What is this distance in kilometers?

The length of the wingspan of an airplane is 40 m.
21. Find the wingspan of this airplane in centimeters.
22. Find the wingspan of this airplane in kilometers.

A compact disc has a diameter of 120 millimeters.
23. What is the diameter of a compact disc in centimeters?
24. What is the diameter of a compact disc in decimeters?

25. Ying and Carol jog exactly 2 miles together every Saturday.
They both guess how many inches their jog is.
Ying guessed 100,000 inches and Carol guessed 10,000 inches.
Whose guess is closer to the actual amount?

Micrometers are metric units that can be used to measure very small objects. In fact, 1 millimeter is equal to 1000 micrometers.
26. How many micrometers are in 0.25 millimeters?
27. How many micrometers are in 1 kilometer?

Now try these:
Lesson 4.3.1 additional questions — p448

Round Up

So to convert something from one unit to another, you need to make the number either bigger (if you're converting to a smaller unit) or smaller (if you're converting to a bigger unit). Conversion factors are useful, but you have to use them sensibly. There's more on conversions next Lesson.

Lesson 4.3.2

Conversions and Proportions

California Standard:
Algebra and Functions 2.1
Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

What it means for you:
You'll convert measurements from one unit to another using proportions.

Key words:
- proportion
- conversion
- ratio

Last Lesson, you saw that to convert between different units you can multiply or divide by a conversion factor. But you can also think about conversion factors as ratios. And where there are ratios, proportions can't be far behind.

This Lesson is all about doing conversions using proportions.

A Conversion Table Is a Set of Ratios

A **ratio** is a way of **comparing** two quantities. But you've seen that ratios can also be used for **converting** quantities from one measuring system to another (think back to scale drawings, for example, where you saw things like "1 centimeter represents 10 meters").

In fact, you can think of the conversion tables you saw last Lesson as a table of **ratios**. For example, you can say the ratio of inches to feet is **12 : 1**.

Example 1

What is the ratio of:
(i) feet to yards? (ii) yards to feet?

Solution
There are **3 feet in a yard**.

(i) This means the ratio of feet to yards is **3 : 1**.

(ii) Remember... the **order** of the quantities in a ratio is important. If the ratio of feet to yards is 3 : 1, then the ratio of yards to feet must be **1 : 3**.

Don't forget:
You could also write these ratios as "3 feet : 1 yard," "1 m : 100 cm," and so on.

Example 2

What is the ratio of:
(i) meters to centimeters? (ii) centimeters to meters?

Solution
There are 100 centimeters in a meter.
(i) The ratio of meters to centimeters is **1 : 100**.
(ii) The ratio of centimeters to meters is **100 : 1**.

Guided Practice

What is the ratio of:

1. meters to kilometers?
2. kilometers to meters?
3. inches to yards?
4. yards to inches?
5. millimeters to centimeters?
6. centimeters to millimeters?
7. miles to feet?
8. feet to miles?

You Can Use Proportions to Convert Between Units

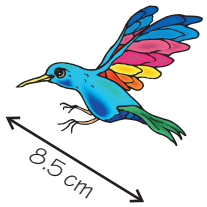
You can use **proportions** to solve problems involving conversions.

The method is exactly the same as the method you've seen in earlier Lessons. You find two equivalent ratios, write a proportion, then solve it using **cross-multiplication**.

Check it out:
The advantage of using proportions rather than multiplying or dividing by a conversion factor is that the method is exactly the same whether you're converting to a smaller unit or a larger unit.

Example 3

The length of a bird is 8.5 cm. Use a proportion to find the length of the bird in millimeters.



Solution

You need two ratios to write a proportion.

- The first ratio is the ratio of **centimeters to millimeters**.
 This is **1 : 10**, or $\frac{1}{10}$.
- The second ratio involves the length of the bird.
 The length in centimeters is 8.5 cm.
 Call its length in millimeters d.
 Then your second ratio is **8.5 : d**, or $\frac{8.5}{d}$.

Don't forget:
You must write both ratios the same way around — either both cm : mm, or both mm : cm.

Now you can write and solve your proportion: $\frac{1}{10} = \frac{8.5}{d}$

$d \times 1 = 8.5 \times 10$ **Cross-multiply**
$d = 85$ **Simplify**

This means that the bird is **85 mm long**.

Check it out:
Notice that the effect is that you multiply the number of centimeters by 10 — just like in the previous Lesson.

Use **exactly** the same method for converting millimeters to centimeters.

Example 4

Convert 125.7 mm to centimeters.

Solution

As always, find two ratios.

- The first ratio is the ratio of **cm : mm**, which is $\frac{1}{10}$.
- The second ratio involves the length you're converting.
 Call the length in centimeters d.
 Then your second ratio is **d : 125.7**, or $\frac{d}{125.7}$.

Check it out:
This method is an alternative method to the method you used last Lesson, but you'll see that you end up with exactly the same answers.

Now write and solve a proportion: $\frac{1}{10} = \frac{d}{125.7}$

$d \times 10 = 1 \times 125.7$ **Cross-multiply**
$10d = 125.7$ **Simplify**
$d = 12.57$ **Divide both sides of the equation by 10**

So 125.7 mm = **12.57 cm**.

Guided Practice

Use proportions to carry out the conversions in Exercises 9–11.
9. What is 48 inches in feet?
10. What is 500 km in centimeters?
11. Convert 14 inches into feet.

In Examples 3 and 4, the ratios were written **without units**.
But if you prefer, you can include units in your ratios, just like you saw with scale drawings. The method works exactly the same.

Example 5

Convert 125.7 mm to centimeters.

Solution

Your first ratio is the ratio of centimeters to millimeters: $\frac{1\text{ cm}}{10\text{ mm}}$

Call the distance you need to find d.

Then your second ratio is: $\frac{d}{125.7\text{ mm}}$

This gives you a proportion: $\frac{1\text{ cm}}{10\text{ mm}} = \frac{d}{125.7\text{ mm}}$

Solve by **cross-multiplication** in the usual way.

$d \times 10\text{ mm} = 125.7\text{ mm} \times 1\text{ cm}$ **Cross-multiply**
$d \times 10 = 125.7 \times 1\text{ cm}$ **Divide both sides of the equation by "mm"**
$10d = 125.7\text{ cm}$ **Simplify**
$d = 12.57\text{ cm}$ **Divide both sides of the equation by 10**

Check it out:
You could even use $\frac{d\text{ cm}}{125.7\text{ mm}}$ here as your second ratio.
Then when you write your proportion and cross-multiply, you get:

$$\frac{1\text{ cm}}{10\text{ mm}} = \frac{d\text{ cm}}{125.7\text{ mm}}$$

or

$$d\text{ cm} \times 10\text{ mm} = 1\text{ cm} \times 125.7\text{ mm}$$

Now you can divide both sides by "cm" and "mm" to find: $d \times 10 = 1 \times 125.7$, or $d = 12.57$
The only difference is that you've canceled the units and so you have to add them back in when you write your final answer.
The point is... as long as you don't do something really bad (like write one ratio with cm on top and the other with mm on top, or forget how to cross-multiply), you should get the correct answer.

Guided Practice

12. What is the ratio of feet to miles?
13. Convert 9000 feet into miles using your ratio from Exercise 12.

Independent Practice

1. What is the ratio of yards to miles?
2. What is the ratio of miles to yards?

3. What is the ratio of centimeters to meters?
4. What is the ratio of meters to centimeters?

Use proportions to find the answers to Exercises 5–8.
5. Convert 7515 yards to miles.
6. Find 0.006 kilometers in millimeters.
7. Jonny needs 69 yards of fencing for his garden. What is this in feet?
8. An Egyptian camel trek is 8.75 km. How far is this in meters?

Now try these:
Lesson 4.3.2 additional questions — p448

Round Up

In this Lesson, you've learned to convert between units using proportions. You can use either method from the last two Lessons to solve conversion problems — you should get the same answer.

Lesson 4.3.3

Converting Between Unit Systems

California Standard:

Algebra and Functions 2.1

Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

What it means for you:

You'll convert between the customary and metric unit systems.

Key words:

- customary system
- metric system
- conversion

So far, you've converted metric units to other metric units, and customary units to other customary units. This Lesson, you'll use conversion tables for converting between the two unit systems. You'll see that all the techniques from the previous Lessons work in exactly the same way in this Lesson too.

Use the Conversion Tables to Help You

The tables below show conversion factors you can use to convert between customary and metric units.

Customary to Metric	
1 inch (in.)	= 2.54 cm
1 foot (ft)	= 30.48 cm
1 yard (yd)	= 0.91 m
1 mile (mi)	= 1.61 km

Metric to Customary	
1 cm	= 0.39 inches (in.)
1 cm	= 0.033 feet (ft)
1 m	= 1.09 yards (yd)
1 km	= 0.62 miles (mi)

You can convert feet to centimeters by **multiplying by 30.48** (to give you a bigger number, since feet are bigger than centimeters).
But you can also convert feet to centimeters by **dividing by 0.033** (which also gives you a bigger number, since 0.033 is less than 1).

So you only really need one of the above tables.
If you know the conversion factors in one table, you can use them to convert from metric to customary **or** from customary to metric.

Check it out:

Notice that the numbers in one table are the reciprocals of the numbers in the other table.

Don't forget:

Dividing by a number gives the same result as multiplying by its reciprocal.
And multiplying by a number gives the same result as dividing by its reciprocal.
See Lesson 3.3.1 for more information.

Example 1

Convert 10 km to miles.

Solution

There are two ways you could use the tables to get your answer.

(i) **Multiply by 0.62** (to get a smaller number, since miles are a bigger unit than kilometers). So 10 km = 10 × 0.62 miles = **6.2 miles**.

(ii) **Divide by 1.61**.
So 10 km = 10 ÷ 1.61 miles = **6.21 miles** (to 2 decimal places)

Check it out:

This may seem more complicated, but there's no new math here. The only difference between this stuff and the calculations between units from the past two Lessons, is that you're using a different conversion table.

Example 2

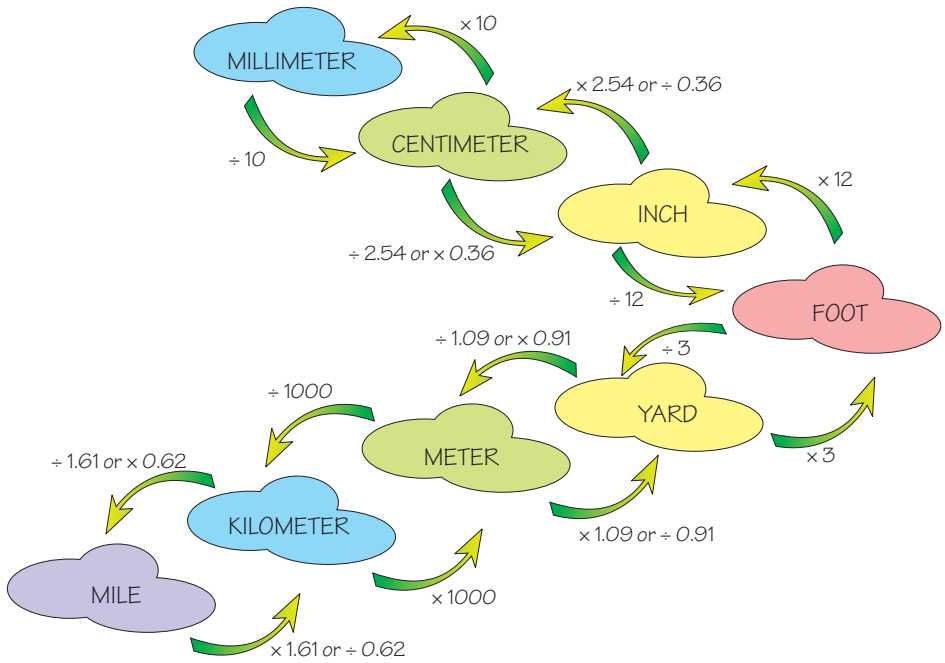
Convert 1 mile to meters.

Solution

From the table, you can see that 1 mile is 1.61 km.
And you've already seen that to convert kilometers to meters you multiply by 1000.

So 1 mile = 1.61 × 1000 meters = **1610 meters**.

The **diagram below** might be helpful for some conversion questions.



Most of the conversion factors for converting between metric and customary systems are only approximations. Most of them are only given to two decimal places.

This means your answer won't always be exact. You can sometimes get **slightly** different answers if you do the question in different ways.

Example 3

Convert 1 yard into meters by:
(i) converting yards to inches, inches to centimeters, and then centimeters to meters,
(ii) using the conversion factor 0.91,
(iii) using the conversion factor 1.09.

Comment on your answers.

Solution

(i) Do the conversion in three stages.
1) yards to inches: 1 yard = **36 inches**
2) inches to cm: 36 inches = 36 × 2.54 cm = **91.44 cm**
3) centimeters to meters: 91.44 cm = 91.44 ÷ 100 m = **0.9144 m**

(ii) 1 meter is slightly bigger than a yard, so multiply by 0.91 to make your number smaller.
1 yard = 1 × 0.91 m = **0.91 m**

(iii) Divide by 1.09 to make the number bigger.
1 yard = 1 ÷ 1.09 m = **0.91743... m**

The answers are slightly different. This is because the conversion factors used are only approximations.

Check it out:
1 inch = 2.54 cm is actually an exact conversion, so 0.9144 meters is the correct answer in this Example.

Check it out:
The various answers in Example 3 are different, but only slightly different.

Guided Practice

Convert the following:

1. 6 meters to yards
2. 18 yards to meters
3. 3 feet to centimeters
4. 2 miles to meters
5. 2 kilometers to yards
6. 6 inches to millimeters
7. 9 yards to centimeters
8. 24 feet to dekameters

9. Josh runs a marathon of 26 miles and 385 yards. How far is this in kilometers?

10. A boat race is 4 miles and 374 yards. How long is the race in meters?

Don't forget:
Depending on how you do these conversions, you might get slightly different answers.

You Can Use Proportions Too

The examples so far have been done by reasoning whether to multiply or divide by the **conversion factor**.

But you could do them using **proportions** if you prefer. Just use the numbers in the conversion tables as ratios.

Example 4

Convert 10 km to miles.

Solution

This is the same as Example 1, but this time it's done with proportions.

As always when using proportions, you need two ratios.

- The first ratio of miles to kilometers comes from the table.
 This is **1 : 1.61**, or $\frac{1}{1.61}$.
- The second ratio involves the measurement you want to convert. Call the converted distance d miles. Then your second ratio is $d : 10$, or $\frac{d}{10}$.

Now write a proportion and solve by cross-multiplication: $\frac{1}{1.61} = \frac{d}{10}$.

$1.61d = 10$. **Cross-multiply**
$d = 10 \div 1.61$ **Divide both sides by 1.61**
$d =$ **6.21 miles** (to 2 decimal places)

Don't forget:
You could include units in your ratios too, if you prefer. See Lesson 4.3.2 for some examples.

Guided Practice

Convert the following using proportions:

11. 10 meters to yards
12. 3 yards to meters
13. 3.7 feet to centimeters
14. 5.1 miles to meters

Convert Weight and Time in Exactly the Same Way

So far, you've only looked at converting **lengths**. But you can convert time and weight (and most other kinds of quantities) in exactly the same way.

You just need to know the conversion factor.

Example 5

Convert 1 hour to seconds.

Solution

Do this in two stages — hours to minutes, and then minutes to seconds.

There are 60 minutes in an hour, and 60 seconds in a minute, so the conversion factor for both stages is 60.

1) Minutes are a smaller unit than hours, so multiply by 60 to get a bigger number: 1 hour = 1 × 60 = **60 minutes**

2) Seconds are a smaller unit than minutes, so again multiply by 60: 60 minutes = 60 × 60 = **3600 seconds**

Guided Practice

Use the information below to convert the quantities that follow.

1 kg = 2.2 lb 1 gallon = 3.79 liters

15. 3 kg to lb **16.** 16 lb to kilograms

17. 7 gallons to liters **18.** 44 liters to gallons

Independent Practice

Don't forget:
Depending on how you do these conversions, you might get slightly different answers.

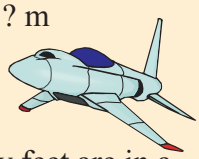
Now try these:
Lesson 4.3.3 additional questions — p449

In Exercises 1–6, find the missing length.
Give your answers to 2 decimal places.

1. 4.5 in. = ? cm **2.** 100 mm = ? in. **3.** 14 yd = ? m

4. 0.5 m = ? in. **5.** 500 cm = ? yd **6.** 45 ft = ? m

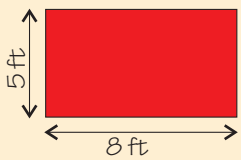
7. The length of a model plane is 56 inches. How long is the model in meters?



8. Michaela and Ricky are each trying to guess how many feet are in a kilometer. Michaela guessed 3000 and Ricky guessed 3500. Whose guess was closest to the correct number of feet?

9. How many kilometers are there in 215,820 inches?

10. The dimensions of Zak's bedroom are 5 ft × 8 ft. What are the dimensions of his room in meters?



Round Up

You've now seen how to convert between different unit systems of length. And if you can do that, you can also convert pretty much anything else you want. For example, you might want to convert dollars to some other currency if you travel overseas.

Lesson 4.3.4

Other Conversions

California Standard:

Algebra and Functions 2.1

Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

What it means for you:

You'll see how to convert between different units where you can't use a simple conversion factor. There's also some information on using calculators.

Key words:

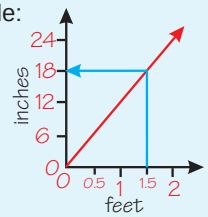
- conversion
- unit
- conversion table
- proportional

Check it out:

For all the conversions you've seen in previous Lessons, zero using one scale (inches, say) was always zero using the other scale too. This is why the first three graphs go through the point (0, 0). For example, 0 km = 0 miles, 0 cm = 0 km, and so on.

Check it out:

You can use this kind of graph to do conversions. You find a length on one scale (1.5 feet, say). Then you go up to the line and across to the other scale. For example: 1.5 feet is the same as 18 inches.



You've seen now how to convert all sorts of things between different units. And you've seen two different techniques for carrying out these conversions — multiplying or dividing by a conversion factor, or solving a proportion.
But these methods don't work for all conversions, unfortunately.

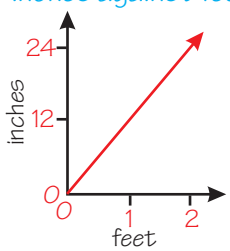
Some Units Are More Complicated to Convert

The methods you've seen so far will work for most conversions you might meet, but **not all** of them.

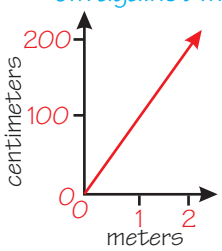
For example, converting **degrees Fahrenheit** (**°F**) to **degrees Celsius** (**°C**) is a little more complicated. Looking at some graphs should help you see the difference.

These first three graphs show various quantities you've already seen.

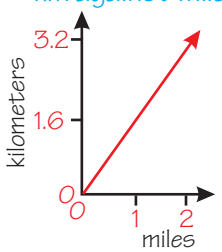
inches against feet



cm against m



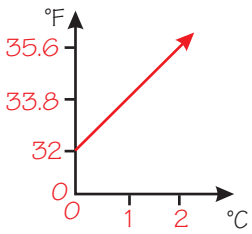
km against miles



They are all slightly different, but they are all **straight lines** and they all go through the **origin**, the point **(0, 0)**. The math term for this is to say that the quantities on each graph are **proportional**.

Now look at the graph showing degrees Fahrenheit (°F) and degrees Celsius (°C).

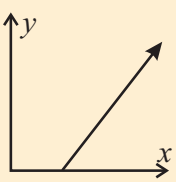
This graph is a straight line, but it **doesn't** go through the point (0, 0). And this means you **can't** use a simple conversion factor, or proportions, to do the conversion.
In other words, °F and °C are **not proportional**.



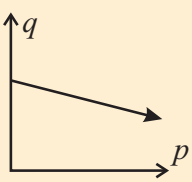
Guided Practice

Using the graphs in Exercises 1–4, say which pairs of quantities are proportional.

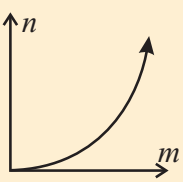
1.



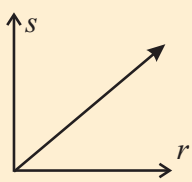
2.



3.



4.



Check it out:
You don't have to learn the formulas for converting Fahrenheit to Celsius, and vice versa.

To convert Celsius to Fahrenheit (or Fahrenheit to Celsius), you have to use a formula.

If F is a temperature in Fahrenheit, and C is a temperature in Celsius, then:

$$F = 1.8C + 32$$
$$C = (F - 32) \div 1.8$$

Notice how you don't just multiply or divide by a conversion factor.
The formula for F involves a multiplication **and an addition**.
The formula for C involves a division **and a subtraction**.

Check it out:
You can write the conversion factors you've seen before as a formula. For example, if m is a length in miles and y is the same length in yards, then $y = 1760m$, where 1760 is the conversion factor.

Example 1

Convert 98.6 °F to degrees Celsius.

Solution

The formula you need to use is: $C = (F - 32) \div 1.8$.
Substitute in your value for F: $C = (98.6 - 32) \div 1.8$.

When you have a formula with more than one operation, you do anything in parentheses first.

So $C = (98.6 - 32) \div 1.8$
$= 66.6 \div 1.8$
$= 37$

This means that **98.6 °F is equal to 37 °C**.

Check it out:
Remember PEMDAS in Lesson 2.1.4.
1. **P**arentheses
2. **E**xponents
3. **M**ultiplication / **D**ivision
4. **A**ddition / **S**ubtraction

Example 2

Convert 240 °C to degrees Fahrenheit.

Solution

The formula you need to use is: $F = 1.8C + 32$.
Substitute in your value for C: $F = 1.8 \times 240 + 32$.

When you have a formula with multiplication and addition, you do the multiplication first.

So $F = (1.8 \times 240) + 32$
$= 432 + 32$
$= 464$

This means that **240 °C is equal to 464 °F**.

Guided Practice

Convert the following:

5. 32 °F to °C
6. 100 °C to °F
7. 50 °F to °C
8. –40 °C to °F
9. 45.2 °C to °F
10. –428.8 °F to °C

The final part of this Lesson (about calculators) is relevant to everything in this book — not just converting between units.

Be Careful If You're Using a Calculator

You can use a calculator for expressions like the ones on the previous page. However, you have to be really careful — different calculators do operations in different orders.

Don't forget:
You have to work out the things in parentheses first. See Lesson 2.1.4 for more information.

For example, if you press the buttons shown on the right, some calculators will do operations **in the order you type them** (and give the answer 50). But other calculators do operations **in the PEMDAS order** (and give the answer 18).

8 + 2 × 5 =

When in doubt about what order your calculator will do operations, **use parentheses buttons** to force your calculator to do what you want.

Example 3

Show how to use parentheses to find 12.5 + 8.2 × 8 on a calculator.

Solution

1 2 · 5 + (8 · 2 × 8) =

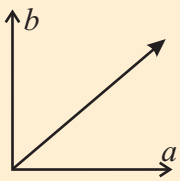
Don't forget:
To enter a negative number, –11 for example, you normally need to press (–) 1 1. But on some calculators you need to press 1 1 +/–.

Make sure you know how to do this on your calculator.

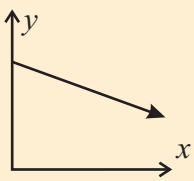
Independent Practice

Which of these pairs of quantities on these graphs are proportional?

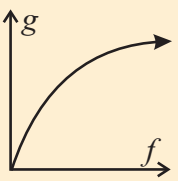
1.



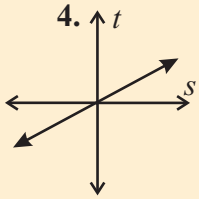
2.



3.



4.



Convert the following:

5. 1445 °F to °C
6. 855 °C to °F
7. 111 °C to °F
8. 16 °F to °C

Evaluate Exercises 9–14 using a calculator.

9. $8g + 3.3$, when $g = 9.2$
10. $52 \times 7 + f$, when $f = 0.251$
11. $1 - 11(k + 2.5)$, when $k = 1.7$
12. $24 - (-8 + 6m)$, when $m = 3$
13. $6n \times (11 + 7n) + 2n$, when $n = 4$
14. $-12.6h - (-11.9h - h^2)^2$, when $h = -3$

15. Joseph tries to calculate 18 ÷ 9 + 4 × 4 on a borrowed calculator. He types 1 8 ÷ 9 + 4 × 4. Why can he not be sure that the calculator will give the correct result? What should he have typed?

Now try these:
Lesson 4.3.4 additional questions — p449

Round Up

This Lesson covered the idea of proportionality, which shows when you can use the conversion methods from previous Lessons. It also reviewed some of the work from earlier in the book about substituting values into formulas, and how to evaluate formulas using a calculator.

Section 4.4 introduction — an exploration into:

Running Rates

You're going to work out how fast the different members of your group can move — either walking, running, hopping, skipping, or crawling. Then you're going to work out how far they could go in different amounts of time, or how long it would take them to cover a particular distance.

Work in groups of 4 or 5. Each person chooses a **different** method to move from place to place. You can choose from: **walking**, **running**, **hopping**, **skipping**, or **crawling**.

Now measure and mark out distances of:
(i) **20 feet**, (ii) **50 feet**, (iii) **70 feet**.
Each person is going to cover **all three** distances in their chosen way, while someone **times** them. Record your results in a chart like this one.

Person	Method	Time in seconds taken to move: 20 feet	50 feet	70 feet
Tim	Running	2.1	5.6	8.0
Carleen	Walking	3.4	8.5	12.0

After each person's 20-foot "run," predict how long their 50-foot and 70-foot "runs" will take.

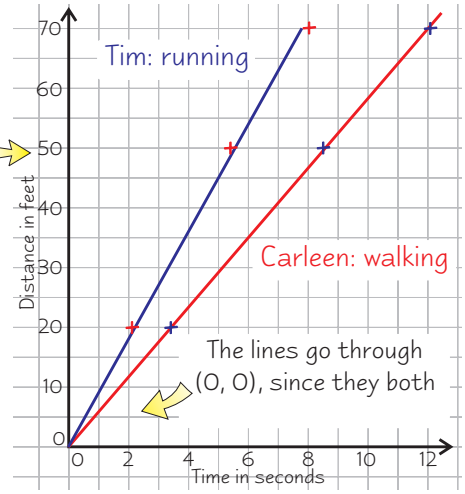
Now you can use your results to make a **graph**.

Example

Plot the results from the table above, and draw best-fit lines.

Label each line clearly with the **person's name** and their **method of traveling**.

Who traveled faster?
How can you tell this from the graph?



Solution

For Tim, plot the three points (2.1, 20), (5.6, 50), (8.0, 70).
For Carleen, plot the points (3.4, 20), (8.5, 50), (12, 70).

Then draw best-fit lines for each person through **(0, 0)** and the three points.

Using the graph, you can see that **Tim traveled faster, as his line is steeper**.

Exercises

1. Use your graph to answer the following questions about the people in your group.
 - **a.** Who moved the fastest and who moved the slowest? How does your graph show this?
 - **b.** Use your graph to predict how long each person would need to go 30 feet.
 - **c.** How far could each person go in 6 seconds?
2. After each person in your group had done their first "runs" of 20 feet, how did you make your predictions about the time they would need to go 50 feet and 70 feet?

Round Up

Distance, time, and speed are closely linked. The slope of your graphs showed speed, and you could use those graphs to work out how long someone would need to cover a certain distance, or how far they would get in a particular time. There's more about speed in this Section...

Lesson 4.4.1

Section 4.4 Rates

You've already seen ratios in Section 4.1. A rate is like a ratio in some ways — it compares two quantities. But unlike ratios, rates have units too — this is the big difference between ratios and rates.

California Standard:

Algebra and Functions 2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.

What it means for you:

You'll learn about what rates are, and how you can use them to compare two quantities.

Key words:

- rate
- per
- ratio

Check it out:

The word "per" is a division word. It tells you to divide the thing before it by the thing after it.

Check it out:

If you walked 8 miles in 2 hours, you could say your rate is "8 miles per 2 hours." However, it's far more useful (and usual) to say that your rate is "4 miles per hour" — this is an example of a unit rate.
Unit rates are probably the most useful kind of rates. They tell you how much one quantity changes when the other changes by one unit (1 hour, say).
Unit rates are easier to compare. For example, it's clear that "8 miles per hour" is faster than "7 miles per hour." But it's not so obvious that "64 miles per 8 hours" is faster than "35 miles per 5 hours."

A Rate Compares Two Quantities

In Section 4.1, you saw that a **ratio** compares two quantities, and you formed a ratio by **dividing**.

A **rate** is a special kind of ratio. It also **compares two quantities**, and it's also formed by **dividing**. But a rate always has **units**, because the quantities you divide are measured in different units.

Speed is a good example of a **rate**. It's often measured in units of "meters per second" or "miles per hour."

A **rate** can be written as a **whole number**, a **decimal**, or a **fraction**.

Example 1



Mike wants to know the rate at which his pinwheel turns in a gentle wind. He counts that it does 18 turns in 9 seconds. What rate is the pinwheel turning at?

Solution

To find the **rate** that the pinwheel turns at, you need to **divide** the **number of turns** it completes by the **time** that it does them in.

$$\text{Rate} = 18 \text{ turns} \div 9 \text{ seconds}$$

$$= \frac{18}{9} \text{ turns per second}$$

$$= \textbf{2 turns per second (or 2 turns/second)}$$

The "/" is pronounced "per". It means "each."

Rates are useful for comparing **how quickly** something is happening.

Example 2

Nicole read 25 pages in 50 minutes, while Santos read 20 pages in 32 minutes. Who read faster?

Solution

First work out both of their reading rates.
Nicole: 25 pages ÷ 50 minutes = **0.5 pages/minute**
Santos: 20 pages ÷ 32 pages = **0.625 pages/minute**

You can see that Santos read **more pages per minute**.
So **Santos read faster than Nicole**.

Guided Practice

Find each of the specified rates in Exercises 1–3.

1. Find the "number of gallons of water per tank" if there are 25 gallons in 5 tanks.

2. Find the "distance traveled per gallon of fuel" if a car travels 120 miles using 8 gallons of fuel.

3. Find the "distance traveled per minute" if a cyclist travels 3 km in 10 minutes.

Vikram paid \$2.44 for four pens. Bob paid \$3.90 for six pens.

4. Calculate "how many cents per pen" Vikram paid.

5. Calculate "how many cents per pen" Bob paid.

6. Who got the better deal, assuming all the pens are identical?

The Rate Depends on Which Way Around You Divide

When you work out a **rate**, you always divide one quantity by another. This means you can make two different rates using the same quantities. It depends which way around you do the division.

Check it out:

You can do something similar using speed. If you took 4 hours to walk 8 miles:

- The rate you walked at would be $\frac{8}{4}$ miles per hour, or 2 miles/hour.
- But you could also say that the rate you walked at was $\frac{4}{8}$ hours per mile, or 0.5 hours/mile.

Both answers are correct. One tells you how far you walk in each hour, the other tells you how long it takes to walk each mile.

Example 3

There are US \$9.40, to every British £5.
Work out the exchange rate in:

(i) the number of dollars per pound

(ii) the number of pounds per dollar

Solution

(i) To work out the **exchange rate** in **dollars per pound**, divide the number of dollars by the number of pounds.

$$\text{dollars per pound} = \frac{\$9.40}{£5} = \mathbf{1.88\ \$/£}$$

You do the same to the **units** as to the **numbers**. Here, you divided dollars by pounds, so the units of the rate are **dollars per pound (\$/£).**

(ii) This time, you need the exchange rate in **pounds per dollar**.
So divide the number of pounds by the number of dollars.

$$\text{pounds per dollar} = \frac{£5}{\$9.40} = \mathbf{0.53\ £/\$}$$

Notice that this time the units are **pounds per dollar (£/\$).**

Check it out:

"Weight per unit cost" means you divide the weight by the cost to find the weight per dollar, or weight per cent.

Example 4

At the movie theater, Sanjay buys a 15 oz bag of popcorn for \$4. Find:

(i) the number of ounces of popcorn per dollar,

(ii) the number of dollars per ounce of popcorn.

Solution

These are the two ways of expressing this rate.

(i) weight per unit cost: 15 oz ÷ \$4 = **3.75 oz/\$**

(ii) cost per unit weight: \$4 ÷ 15 oz = **0.27 \$/oz (to 2 decimal places)**

Don't forget:

The two different rates must be reciprocals. You use the same two quantities to work them out, but the numerator of one rate is the denominator of the other, and vice versa. For example:

pounds per dollar = $\frac{\text{pounds}}{\text{dollars}}$

but

dollars per pound = $\frac{\text{dollars}}{\text{pounds}}$

There's more information about reciprocals in Lesson 3.3.1.

The **two ways** of **expressing a rate** are **reciprocals** of one another.

For example $\frac{1}{1.88\ \$/£} = 0.53\ £/\$$, and $\frac{1}{0.53\ £/\$} = 1.88\ \$/£$

Notice how taking a reciprocal of a quantity **swaps its units around** too.

Example 5

Car A does 18 miles per gallon, while Car B does 0.05 gallons per mile. Which car is more efficient?

Solution

To convert **gallons per mile** into **miles per gallon**, form the reciprocal.

0.05 gallons per mile = $\frac{1}{0.05}$ miles per gallon, or **20 miles per gallon**

Now you can **compare** the two quantities:

Car B is more efficient, as it drove more miles per gallon of fuel used.

Guided Practice

Check it out:

If time is involved in a rate, it is usually the denominator. Think about it — miles/hour, words/minute, $/hour.
Also, if you are comparing two times (for example the number of hours you are at school every week), the larger unit usually goes on the bottom — hours/week, for example.

Find the rates in Exercises 7–10.
Explain what each of the quantities actually means.

7. 27 boats ÷ 3 lakes
8. 10 meters ÷ 10 seconds
9. 16 oranges ÷ 8 people
10. 150 words ÷ 3 minutes

Express the rates in Exercises 11–14 in two different ways.
You must include units in each of your answers.

11. A woman uses 2 gallons of gas driving 50 miles.
12. A store sells 5 ounces of parsley for $2.
13. A man takes 1 hour to drive 30 miles.
14. A rabbit hops 10 meters in 4 seconds.

Independent Practice

Now try these:

Lesson 4.4.1 additional questions — p449

In Exercises 1–4, work out two unit rates using the given quantities.

1. 12 ducks on 3 ponds.
2. 150 gallons in 50 seconds.
3. 5 apples between 2 people.
4. 1360 km in 20 hours.

In Exercises 5–8, say what units the rates will have.

5. 4 miles ÷ 8 minutes.
6. 3 dollars ÷ 7 ounces.
7. 2 liters ÷ 5 glasses.
8. 21 carrots ÷ 7 rabbits.

9. Kevin walked 6 feet in 4 seconds, while Aretha walked 10 feet in 5 seconds. Who was walking faster?

10. Cedro bought 10 ounces of Cereal A for $2.25. Then he bought 15 ounces of Cereal B for $3.30. Which cereal is less expensive?

Round Up

Both rates and ratios involve dividing one quantity by another, but with rates you have to remember to include the units as well. You see rates all the time in real life — they're incredibly important.

Lesson 4.4.2

Using Rates

In the last Lesson, you saw how to form a rate by dividing two quantities. In this Lesson, you're going to see how you can use a rate to work out what one of those two quantities must have been.

It's like doing the last Lesson again, only backward.

California Standards:

Algebra and Functions 2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.

Algebra and Functions 2.3 Solve problems involving rates, average speed, distance, and time.

What it means for you:
You'll learn about how to use rates to find other quantities.

Key words:
- rate
- per

Multiply by the Rate to Find One of the Quantities

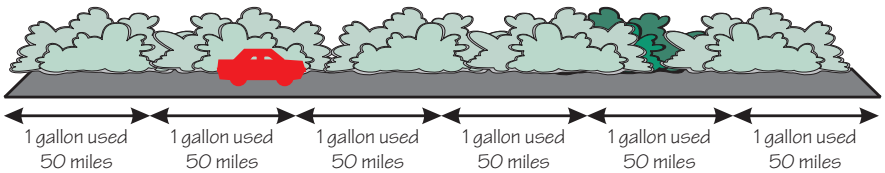
A **rate** explains the **connection** between two quantities. So if you know a rate and **one** of the quantities, you can work out the value of the **other**.

For example, if you know that a car can travel 150 miles on 3 gallons of fuel, you can work out its fuel efficiency in **miles per gallon**.

$$\textbf{Miles per gallon} = \frac{\text{Miles traveled}}{\text{Gallons used}} = \frac{150 \text{ miles}}{3 \text{ gallons}} = \textbf{50 miles per gallon}$$

Now suppose the same car did another journey, and used **6 gallons** of fuel.

For every 1 gallon of fuel, it would travel 50 miles.
So if it used 6 gallons of fuel, it must have traveled 6 × 50 = **300 miles**.



$$\textbf{Number of Miles} = \frac{\textbf{Miles}}{\textbf{Gallons}} \times \textbf{Number of Gallons}$$

The same idea works for all rates. If you **multiply** a rate "**X per Y**" by a value of Y, you find the corresponding value of X.

Check it out:
You'll see this idea again in later Lessons in connection with speed.

Check it out:
X per $Y = \dfrac{\text{value of } X}{\text{value of } Y}$.
This means:
value of X
$=$ value of $Y \times (X$ per $Y)$

To see why, multiply both sides of the first equation by the value of Y.

Check it out:
If the rate is in "dollars per ounce," then you multiply by the number of ounces to find the cost in dollars.

Example 1

Thalia goes to the health-food store to buy 10 ounces of granola. Granola costs $0.25 per ounce. How much will the granola cost?

Solution

This rate means: 0.25 dollars per ounce $= \dfrac{\text{number of dollars}}{\text{number of ounces}}$

So to find the cost, you need to **multiply** the **rate** "**$0.25 per oz**" by the **weight** of granola that Thalia is buying **"10 oz"**.

Number of dollars $= 0.25$ dollars/ounce $\times$ 10 ounces
$= 2.5$ dollars

The granola will cost her $2.50.

Check it out:

Remember...
value of X = rate (X per Y) × value of Y

Here, the rate is "days per room," meaning: X is "days," and Y is "rooms."

So you have to multiply the rate by number of rooms to get the time in days.

Example 2

A painter paints at a rate of 2 days per room.
How long will she need to paint 9 rooms?

Solution

This rate means: $\text{2 days per room} = \dfrac{\text{number of days}}{\text{number of rooms}}$

So to find the number of days she will need, **multiply** "**2 days per room**" by the number of rooms.

Number of days $= 2 \text{ days/room} \times 9 \text{ rooms}$
$= \mathbf{18 \text{ days}}$

She will need 18 days.

Guided Practice

1. The math teacher gives out 2 exercise books per student.
If she has 24 students, how many exercise books will she need?

2. The manufacturing cost of baseballs is 32¢ per ball.
How much would it cost to manufacture 88 baseballs?

3. A particular kind of carpet costs $9 per square yard.
What would it cost to carpet a room with a floor area of 16 yd^2?

4. A car travels 11 miles per liter of fuel.
How far will the car be able to travel on 7 liters of fuel?

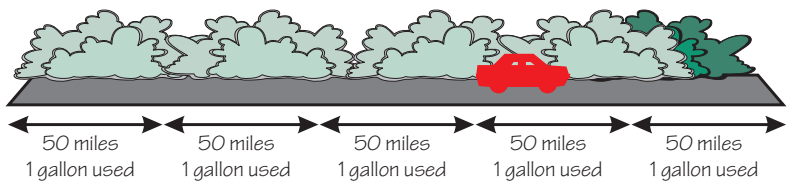
5. An electric heater gives out heat at the rate of 3000 joules per second. If the heater is on for 30 minutes, how many joules of heat will be given out altogether?

Check it out:

For Exercise 4, remember to use the number of seconds, because the rate is given in "joules per second." (A "joule" is a unit of energy.)

Divide by the Rate to Find the Other Quantity

Think back to the example of a car with a fuel efficiency of **50 miles per gallon**. If that car now goes on a journey of **250 miles**, you can use its fuel efficiency to find the amount of **fuel** used.



$$\textbf{Number of Gallons} = \textbf{Number of Miles} \div \frac{\textbf{Miles}}{\textbf{Gallons}}$$

The same idea works for all rates. If you **divide** a value of X by a rate "**X per Y**," you find the corresponding value of Y.

Check it out:

$X \text{ per } Y = \dfrac{\text{value of } X}{\text{value of } Y}$.

This means:

value of Y
$= \text{value of } X \div (X \text{ per } Y)$

To see why, multiply both sides of the first equation by the "value of Y," and then divide by the rate "X per Y."

Check it out:

Remember...
value of Y = value of X
÷ rate (X per Y)

Here, the rate is "dollars per ounce," meaning:
X is "dollars," and
Y is "ounces."

So you have to divide the number of dollars by this rate to find the weight in ounces.

Example 3

Thalia goes to the health-food store with $4.
Granola costs $0.25 per ounce. How much granola can she buy?

Solution

This rate means: 0.25 dollars per ounce $= \dfrac{\text{number of dollars}}{\text{number of ounces}}$

So to find the number of ounces of granola, you need to divide the number of dollars "**$4**" by the **rate** "**$0.25 per oz**."

Number of ounces = 4 dollars ÷ 0.25 dollars/ounce
= 16 ounces

Thalia can buy 16 oz of granola.

Example 4

A painter paints at a rate of 2 days per room.
How many rooms will she be able to paint in 38 days?

Solution

This rate means: 2 days per room $= \dfrac{\text{number of days}}{\text{number of rooms}}$

So to find the number of rooms she can paint, **divide** the number of days by the rate "**2 days per room**."

Number of rooms = 38 days ÷ 2 days/room
= **19 rooms**

She will be able to paint 19 rooms.

Guided Practice

6. The math teacher gives out 3 exercise books per student. If she has 39 books, how many students will she have enough books for?

7. The manufacturing cost of basketballs is $1.14 per ball.
How many balls could be manufactured for $136.80?

8. A particular kind of carpet costs $11 per square yard.
What area of floor could be carpeted for $264?

9. A car travels 12 miles per liter of fuel.
How much fuel would be used on a journey of 192 miles?

10. An electric heater gives out heat at the rate of 3500 joules per second. If 4,200,000 joules of heat have been given out, for how long has the heater been on?

Don't forget:

You can treat units like numbers. If there is a unit on the top line of a fraction, and the same one on the bottom line of it, you can cancel them out, just like numbers.

Check it out:

You need to think carefully about whether you're being asked for the higher or lower unit rate.
Words like "faster" and "most" are usually telling you to look for the higher unit rate.
Words like "slower" and "least" are usually telling you to look for the lower unit rate.

Now try these:

Lesson 4.4.2 additional questions — p450

Independent Practice

1. Joe types at a rate of 150 words per minute. If he has typed 1500 words, how long has he been typing for?

2. Terrence buys some soil for his garden. It costs \$90/m^3. He spends \$225. How many m^3 of soil did he buy?

3. The city bus uses gasoline at a rate of 15 miles/gallon.
How far can the bus go on 3 gallons of fuel?

4. Jen drinks 8 glasses of water per day. How many days will it take her to drink 28 glasses of water?

5. Letitia is selling her 28 baseball cards. Todd offers her \$0.25 per card. Daria offers \$1.50 per 7 cards. Who is offering to pay more?

6. Find the distance traveled per hour if a person runs 8 miles in 2.5 hours.

7. The rate at which water comes out of a faucet is 6 liters per minute. How long would it take to fill a bath with a capacity of 130 liters?

8. Water comes out of a faucet at a rate of 0.85 liters per second. How much water will be supplied if the faucet is left on for 3 minutes?

9. An aircraft is traveling at 900 miles per hour. The pilot sees a mountain when he is 50 miles away. How many minutes will it be before the aircraft reaches the mountain?

10. Mr. Clark's car holds 12 gallons of gasoline when the tank is full. Starting with a full tank, he drove 153 miles, and used 25% of the gas. How many miles to the gallon does his car get?

11. At a health-food store, breakfast cereal is sold by the weight.
If you can normally buy 2 lb of cereal for \$3, then how much could you buy for \$4 if there is a sale and the price of everything in the store is reduced to 50% of its normal price?

12. At 50 miles per hour, a car can travel 600 miles on a full tank of 15 gallons of fuel. At 30 miles per hour, the car can travel 30% further for each gallon of fuel. If the car travels 250 miles at 30 miles per hour, how many gallons of fuel would it use? And how long would the journey take?

Round Up

Now you know how to use a rate to calculate another quantity. But you have to be sure you use the rate in the correct way. In the last Lesson, you saw that rates were important in real life. Well... in the next Lessons, you'll be thinking about one rate in particular — speed. And speed is one of the most common rates you'll ever come across. You might even have to use it every day.

Lesson 4.4.3

Finding Speed

California Standards:
Algebra and Functions 2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.
Algebra and Functions 2.3 Solve problems involving rates, average speed, distance, and time.

What it means for you:
You'll see how to use measurements of distances and times to calculate speeds.

Key words:
- rate
- speed
- distance
- time
- average

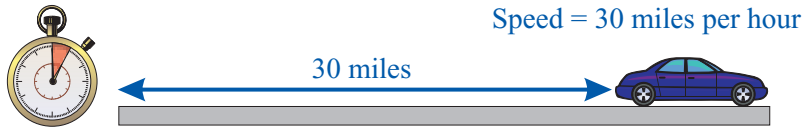
Check it out:
A speed in miles/hour is a measure of the distance that the car would cover if it carried on traveling exactly at that speed for a whole hour.

The rate that you'll come across most often in everyday situations is speed. Speed is a rate because to find it you divide a distance by a time. It's a measure of the rate that an object moves at.

Speed Is a Rate

A rate **compares** two quantities. You've seen in the last two Lessons that to find a rate you **divide** one quantity by another.

Think about a car — you would usually talk about the speed that the car was going in **miles per hour**. That's a **rate**, because you're dividing the number of **miles** the car has gone by the number of **hours** it took.



$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

Use the Formula to Calculate Speeds

You can use the formula **Speed = Distance ÷ Time** to calculate the speed at which an object is traveling.

Example 1

Will walked for 2 hours at a constant speed up a mountain trail that is 6 miles long. What was Will's hiking speed?

Solution

Distance = 6 miles
Time = 2 hours

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{6\text{ miles}}{2\text{ hours}} = \mathbf{3\text{ miles/hour}}$$

Guided Practice

1. Feo's solar-powered lawn mower is set to travel at a constant speed. If it covers 1 mile in 2 hours, what speed is it traveling at?

2. A satellite is orbiting the earth at a constant speed. It takes 2 hours and 20 minutes to complete an orbit of length 56,280 km. What is its speed in km/min?

3. A plane travels at a constant airspeed. It covers a distance of 375 km in 3 hours. What is the plane's airspeed?

4. Light travels at a constant speed. Calculate the speed of light, given that it travels 74,948,114.5 m in 0.25 seconds.

There Are Many Different Units of Speed

There are a number of common units of speed, and some less common ones. To compare speeds, you need them to be in the **same units**.

Example 2

Joshua is driving at a speed of 40 miles/hour. His friend is driving at a speed of 17 yards/second. Who is traveling more quickly?

Solution

To compare the speeds, you need them to be in the same units.
Joshua's friend is driving at 17 yards/second, so he travels 17 yards each second.
So in 1 minute he travels $17 \times 60 = 1020$ yards.
And in 1 hour he travels $1020 \times 60 = 61{,}200$ yards.
So you could say his speed is **61,200 yards per hour**.

But $61{,}200 \text{ yards} = 61{,}200 \div 1760 = 34.77$ miles.
So you could also say his speed is **34.77 miles per hour**.

Now that both speeds have the same units, you can compare them directly.

Joshua is traveling at 40 miles/hour, while his friend is traveling at 34.77 miles/hour, so **Joshua is travelling more quickly.**

Don't forget:

You need to know a few conversion factors for questions like this.
1 mile = 1760 yards
1 mile = 1.61 km
1 km = 1000 meters
1 hour = 60 × 60 = 3600 s.
You saw some of these in Lesson 4.3.3

Guided Practice

The fastest bird in the world is the peregrine falcon.
This bird can reach speeds of 300 kilometers/hour when it swoops.
5. What is this in miles/hour?
6. What is this in meters/second?

The peregrine falcon's cruising speed is 24.6 meters/second.
7. What is this speed in miles per hour?
8. How far could it fly in 3 hours?

9. The fastest fish in the world is the sailfish. Its top speed is about 68 miles/hour. How fast is this in meters/second?

This Formula Gives You the Average Speed

Suppose a car goes on a journey. Its speed will **vary** all the time. It may **stop** at a crosswalk or traffic lights, but then it will go **faster** on open roads.

If you work out the car's **speed** by dividing the total **distance** it went by the **time** it took, you are calculating its **average speed** over the **whole journey**.

This doesn't mean the car was always traveling at that average speed.
It just means that if it did travel at that speed for the journey time, it would cover the same distance as the actual journey.

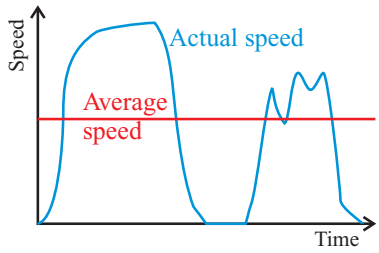
Check it out:

If an object is traveling at a constant speed, then its speed at any moment will be exactly the same as its average speed.

Check it out:
The car's speedometer tells the driver how fast it is traveling at any instant in time — not the average speed.

This graph shows the actual and average speeds for a car journey.

The **blue** line shows **speedometer readings**.
The **red** line shows the **average speed**.



Guided Practice

10. A fish swims from one end of a lake to the other in 2 hours. The lake is 12 km long. What is the average swimming speed of the fish?

Check it out:
There's more information about average speeds in Lesson 4.4.5.

A skydiver jumped out of a plane at 10,800 feet.
He fell for 50 seconds before his parachute opened at 3000 feet.
It took a further 200 seconds for him to reach the ground.
11. What was his average speed with his parachute closed?
12. What was his average speed with his parachute open?
13. What was his average speed over the whole drop?

Independent Practice

1. A sprinter completes the 100 m dash in 10 seconds.
What is his average speed over the course of the race?

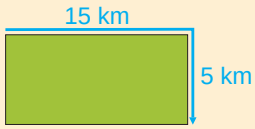
2. Shona is riding her scooter. She covers 4 blocks in 10 minutes.
How fast is she traveling? Give your answer in blocks/minute.

Don't forget:
Always remember that the units of an answer should match the units in the question. For a reminder, see Lesson 4.4.1.

3. A hockey player shoots the puck a distance of 51 m in 1.5 s.
What is the average speed of the puck?

4. A cheetah runs one mile at constant speed. It takes 1 minute.
What is the cheetah's speed in miles per hour?

5. Ryan rides his bike along the edge of the park, taking the route shown in the diagram.
It takes him 4 hours. What is his average speed?



6. A sailing boat travels a distance of 2 miles in 15 minutes.
What is its average speed in meters/second?

Now try these:
Lesson 4.4.3 additional questions — p450

Laura is standing 255 m away from Ervon. When she rings a bell, he hears the sound of the bell 0.75 seconds later.
7. What is the speed of sound in air in meters per second?
8. What is the speed of sound in air in miles per hour?

9. A raindrop forms in a cloud at an altitude of 1750 m.
It takes 250 seconds to fall to Earth. What is its average speed?

Round Up

You can work out an object's speed by dividing the distance it traveled by the time it took. Unless the speed of the object is constant, that speed will be an average speed over the whole journey. In the next Lesson, you'll see how to rearrange the formula that you've learned — and use speeds to work out times and distances.

Lesson 4.4.4

Finding Time And Distance

California Standards:

Algebra and Functions 2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.

Algebra and Functions 2.3 Solve problems involving rates, average speed, distance, and time.

What it means for you:

You'll see how to use speeds to calculate distances and times.

Key words:

- rate
- speed
- distance
- time

Check it out:

A rate that is describing how far an object moves in a certain time is a speed.

Check it out:

This is a rearrangement of the formula "Speed = Distance ÷ Time."

Speed = Distance ÷ Time.

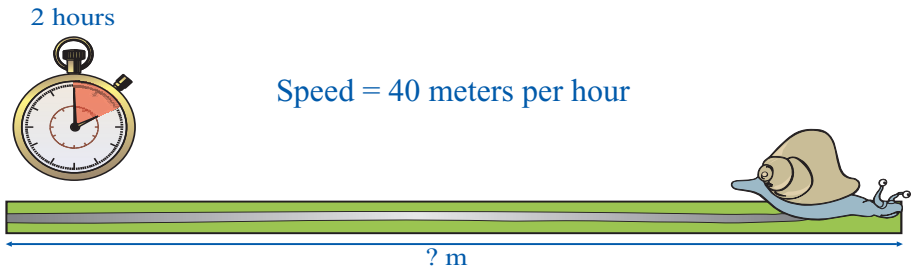
↓ multiply both sides by "Time"

Speed × Time = Distance.

In the last Lesson, you saw how to use the formula "Speed = Distance ÷ Time" to work out the speed of an object. In this Lesson, you'll see how to use the formula to work out the distance an object has traveled or the time it took when you're given its speed. It's really just a "special case" of what you saw in Lesson 4.4.2.

Use Speed and Time to Find Distances

An ordinary garden snail moves at an **average** rate of 40 meters per hour. Suppose you want to find out **how far** it would travel in 2 hours.



In one hour, it will move 40 m. So in two hours it will move $2 \times 40 =$ **80 m**. The **rate** has been **multiplied** by the **time** it took.

If you know the **rate** at which an object is traveling and the **time** that it is traveling for, you can work out the **distance** it travels using this formula:

$$\textit{Distance} = \textit{Speed} \times \textit{Time}$$

Example 1

Mrs. Jackson walks at an average speed of 2 miles per hour. If she walks for 3 hours, how far will she go?

Solution

$$\text{Distance} = \text{Speed} \times \text{Time} = 2\ \frac{\text{miles}}{\text{hours}} \times 3\ \text{hours} = \textbf{6 miles}$$

Example 2

Mr. Jackson walks at an average speed of 0.4 meters per second. If he walks for 9850 seconds, how far will he go?

Solution

$$\text{Distance} = \text{Speed} \times \text{Time} = 0.4\ \frac{\text{meters}}{\text{second}} \times 9850\ \text{seconds} = \textbf{3940 meters}$$

These two Examples show that the units of the **final answer** depend on the units of the other quantities. If you use a speed in **meters per second** and a time in **seconds**, the distance you find would be in **meters**.

Guided Practice

1. Raymon drives his car at an average speed of 40 miles/hour for 3 hours. What distance will he cover?

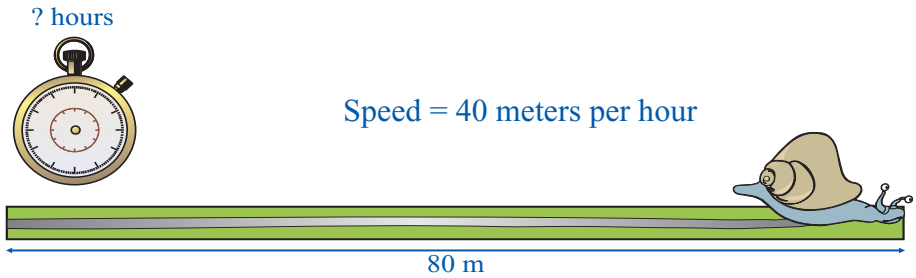
2. Alice wins a race on field day in a time of 50 seconds. Her average speed is 8 m/second. Which length race did she win?

3. A sunflower grows at a rate of half a centimeter a day. How tall will it be after 30 days?

4. A stone falls off a cliff. It hits the ground after 3 seconds. The stone traveled at an average speed of 14.7 m/second. How high is the cliff?

Use Speed and Distance to Find Times

Think about the snail again. This time you know that it's gone 80 meters, but you don't know how long it took.



To find the **time** that it took the snail to travel 80 m, you can **divide** the **distance** it traveled by the **speed** it moves at.
Time taken = 80 ÷ 40 = **2 hours**.

If you know the **speed** at which an object is traveling, and the **time** that it is traveling for, you can work out the **distance** that it travels using this formula:

$$\textit{Time} = \frac{\textit{Distance}}{\textit{Speed}}$$

Check it out:

This is a rearrangement of the formula "Speed = Distance ÷ Time."

Speed = Distance ÷ Time.
↓ multiply both sides by "Time"
Speed × Time = Distance
↓ divide both sides by "Speed"
Time = Distance ÷ Speed

Example 3

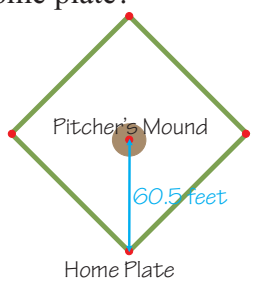
A pro baseball pitcher pitched a baseball at a speed of 151.1 feet per second. How long did it take the ball to travel the 60.5 feet from the pitcher's mound to home plate?

Solution

$$\text{Time} = \frac{\text{distance}}{\text{speed}}$$

$$= 60.5 \text{ feet} \div 151.1 \frac{\text{feet}}{\text{second}}$$

$$= \textbf{0.400 seconds (to 3 decimal places)}$$



Guided Practice

5. A bird flies at an average speed of 22 miles/hour. It covers a distance of 11 miles. How long was it flying for?

6. Eddie's house is 3.4 km away from his school. He cycles to school every morning. His average speed is 0.2 km/minute. How long will it take him to get to school?

7. A caterpillar eats leaves at a rate of 11 leaves/hour. How long will it take it to eat 55 leaves?

8. A train travels a distance of 581 miles at an average speed of 83 miles per hour. How long will it take to do this?

Independent Practice

Don't forget:
"meters per second" is often written as "m/s."

Don't forget:
These average speeds are unit rates. For a reminder on comparing unit rates, see Lesson 4.4.1.

Now try these:
Lesson 4.4.4 additional questions — p450

1. Caryne, Reese, and Katia ran a 100 m race. Caryne's average speed was 8.52 m/s, Reese's was 9.11 m/s, and Katia's was 9.02 m/s. Who won the race?

2. A horse trots at a speed of 9 miles/hour. How much ground will the horse cover in 3 hours?

3. A hot-air balloon goes up at a speed of 50 feet/minute. It rises to a height of 1100 feet. How long will it take to reach this height?

4. Fayard watches a raindrop run down a window. The window is 150 cm tall. The drop runs at an average speed of 2 cm/second. How long will it take to reach the bottom of the window?

5. Dulcie has a model plane that flies at a constant speed of 1.5 m/s. If it is airborne for 30 seconds, how far will it go?

6. A roller-coaster ride lasts 90 seconds. The roller-coaster train travels at an average speed of 50 km/hour. How far does it travel?

7. A train leaves the city at 10 a.m. It travels at an average speed of 93 miles/hour. The next stop is 62 miles away. At what time will it reach the next stop?

8. Town A is 4 miles from Town B. A bus leaves Town A for Town B traveling at an average speed of $\frac{1}{2}$ mile/minute. Mr. Jones lives in Town B. He lives $\frac{1}{4}$ mile from the bus stop, and walks at an average speed of $\frac{1}{28}$ mile/minute. Mr. Jones leaves his house to catch the bus at the same time the bus leaves Town A. Will he catch the bus or miss it?

Round Up

In the same way that you can use a time and a distance to find a speed, you can use a speed to find a time or distance. It's just a case of rearranging the formula and putting the numbers in.

Lesson 4.4.5

Average Rates

You've already seen the formulas linking average speed, distance, and time. In this Lesson, you'll get a lot of practice using them.

California Standards:

Algebra and Functions 2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.

Algebra and Functions 2.3 Solve problems involving rates, average speed, distance, and time.

What it means for you:

You'll see how to work out average speeds over a whole journey.

Key words:

- rate
- speed
- distance
- time
- average

Always Use Your Three Formulas

You've seen three formulas linking time, distance, and speed.

$$Speed = \frac{Distance}{Time} \qquad Time = \frac{Distance}{Speed}$$

$$Distance = Speed \times Time$$

You need to use these formulas for all questions involving time, distance, and speed. Don't try to take any shortcuts.

Example 1

A cyclist travels at 18 miles per hour for one hour, and for another hour at 32 miles per hour. What is her average speed for the whole journey?

Solution

To find her average speed for the whole journey, you need the **total distance** and the **total time**.

The question tells you that the total **time** is **2 hours**.

To find the total **distance**, break the journey into two parts, and use one of the above formulas for each part.
For the first part: distance = 18 miles per hour × 1 hour = **18 miles**.
For the second part: distance = 32 miles per hour × 1 hour = **32 miles**.
So the total distance = 18 + 32 = **50 miles**.

Now find the average speed for the whole journey.
Average speed = 50 miles ÷ 2 hours = **25 miles per hour**

Check it out:

The answer to Example 1 is the average of the two speeds. But you should never assume that things will be that easy. Look at Example 2.

Now look at this next example.

Example 2

Joshua walks for 1 mile at 4 miles per hour, and for another mile at 2 miles per hour. What is his average speed for the whole journey?

Solution

Do the question in two parts again.
This time, the total distance is **2 miles**.

For the first part: time = 1 mile ÷ 4 mph = **0.25 hours**.
For the second part: time = 1 mile ÷ 2 mph = **0.5 hours**.
So the total time = 0.25 + 0.5 = **0.75 hours**.

This means the average speed is 2 ÷ 0.75 = **2.67 miles per hour**.

Check it out:

The answer to Example 2 is not the average of the two speeds. The "obvious" answer of "3 miles per hour" would have been wrong.

Guided Practice

Work out the average speed of the entire journey in Exercises 1–4.

1. Eloise rides her bike for 2 hours. For the first hour, her average speed is 10 km/h. For the second hour, her average speed is 8 km/h.

2. Orlin rows upstream for 20 minutes at a speed of 0.1 km/min. Then for 10 minutes he rows downstream at 0.4 km/min.

3. Vanessa rides on an elephant for 2 hours (average speed 4.5 miles/hour) and a camel for 1 hour (average speed 3 miles/hour).

4. Brian pulls his sled up a hill in 120 s at a speed of 0.6 m/s. Then he slides back down in 30 s at a speed of 2.4 m/s.

Always Rely on Those Same Three Formulas

You can use those formulas to solve some very complex problems. You just have to take things one step at a time.

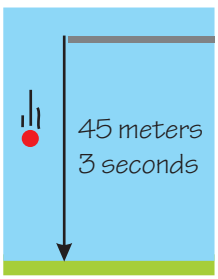
Example 3

A ball is dropped vertically from a platform 45 meters above the ground. It falls for 3 seconds before it hits the ground. The ball then bounces vertically, and rises for 2 seconds at an average speed of 10 m/s.

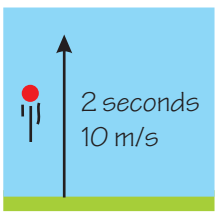
What is the ball's average speed as it falls?
How high does the ball rise?
What is the ball's average speed over the two parts of the journey?

Solution

This looks pretty difficult. Just work **slowly** and **carefully**, and it might also help if you draw some simple diagrams.



For the first part — as the ball is **falling**.
Distance = 45 meters. Time = 3 seconds.
So its average speed as it falls = 45 m ÷ 3 s
= **15 m/s**



For the second part — as the ball is **rising**.
Average speed = 10 m/s. Time = 2 seconds.
So the distance it rises = 10 m/s × 2 s
= **20 m**

To find the average speed for the whole journey, you need the total distance traveled and the total time taken.
Total distance traveled = 45 m + 20 m = **65 m**.
Total time taken = 3 s + 2 s = **5 s**.
So the overall average speed = 65 m ÷ 5 s = **13 m/s**.

Don't forget:

This example looks very complex. But you just have to keep using those same three formulas.

For each part of the journey, there are 3 quantities — distance, time, and speed.

Write down the two quantities you know, then use the formula that will tell you the other one.

Guided Practice

Work out the average speed of the entire journey in Exercises 5–8.

5. A duck swims 50 m at a speed of 5 m/s, then flies 550 m at a speed of 11 m/s.

6. Glover walks 1 mile to the station to catch a train. He walks at 2 miles/hour. He gets the train for 20 miles. Its speed is 80 miles/hour.

7. Diana drives 5 km at a speed of 15 km/h, then 10 km at a speed of 30 km/h.

8. Justin is in a circus parade. He walks 50 m at a speed of 1 m/s. Then he puts on stilts, and walks another 140 m at a speed of 0.7 m/s.

Independent Practice

Work out the average speed of the entire journey in Exercises 1–7.

1. Keshila jogs for 2 hours at a speed of 3 miles/hour, and then walks for 2 hours at a speed of 2 miles/hour.

2. Wayan drives for 100 miles at a speed of 50 miles/hour, then for 50 miles at 25 miles/hour.

3. A farmer drives his combine for 5 hours at an average speed of 15 km/h. Then he drives his tractor for 3 hours at an average speed of 10 km/h.

4. Hannah is doing the long jump. Her run up is 24 m and her run-up speed is 8 m/s. She jumps 6 m. Her jump speed was 6 m/s.

5. A rose grows at a speed of 0.5 mm/day for 4 days. Then it rains. For the next 6 days, the rose grows at the faster speed of 1 mm/day.

6. A spider climbs upward for 90 cm at a speed of 2 cm/s. Then it drops back down 50 cm at a speed of 10 cm/s.

7. Dan competes in a triathlon. He does the 2 km swim at an average speed of $2\frac{2}{3}$ km/h. His 40 km bike ride has an average speed of $22\frac{6}{7}$ km/h. His 10 km run is at an average speed of 20 km/h.

8. A skydiver jumps out of a plane and falls for 43 seconds at an average speed of 190 feet per second before his parachute opens. He then drops the next 2850 feet at 15 feet per second, until he lands. How high was the plane when he jumped, and how long did it take him to reach the ground?

Now try these:
Lesson 4.4.5 additional questions — p451

Check it out:
Even when a journey is split into more than two parts the method is still the same. Work out the time and distance of each part, add them together, and put them into the formula.

Round Up

For nearly all "speed questions," you'll be given two bits of information and you'll need to work out the third using one of the formulas at the beginning of this Lesson. The question might look complicated, and it might be in many parts, but the basic idea will be the same. You use two quantities and one of the formulas to find out the other quantity.

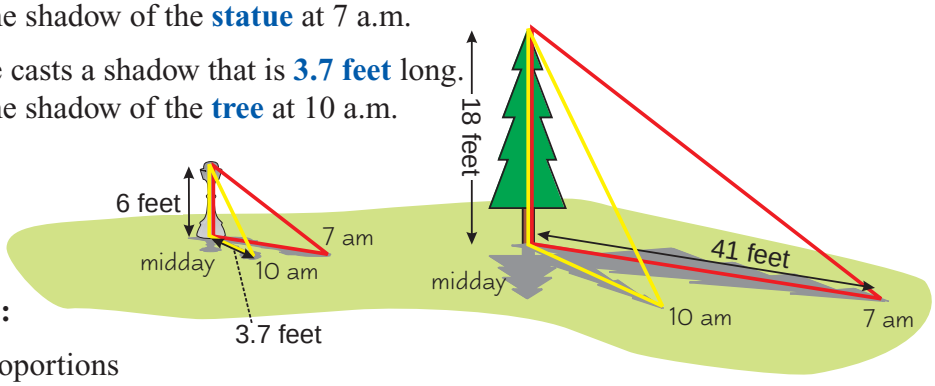
Chapter 4 Investigation
Sunshine and Shadows

Shadows change length and direction as the Sun moves across the sky. The picture below shows a tree and a statue on a sunny day. Their shadows at various times of day are shown.

The two **red triangles** in the diagram are **similar**, as are the two **yellow triangles**.

Part 1:

- At **7 am**, the tree casts a shadow that is **41 feet** long.
 Find the length of the shadow of the **statue** at 7 a.m.
- At **10 am**, the statue casts a shadow that is **3.7 feet** long.
 Find the length of the shadow of the **tree** at 10 a.m.



Things to think about:

- How can you use proportions with similar triangles?

Part 2:

- At **midday**, the shadows are at their shortest.
 The ratio of the **height of the objects** to the **length of their shadows** is **1 : 0.28**.
 Find the lengths of the shadows of both objects at midday.

Extension

- Find the average rate that the length of the statue's shadow shrinks between 7 a.m. and 10 a.m. Give your answer in feet per hour.
- Between the hours of midday and 6 p.m., the tree's shadow increases in length at an average rate of 9.6 feet per hour. Find the length of the tree's shadow at 6 p.m.

Open-ended Extension

- Devise a method for finding the height of a local statue, tree, building, or other landmark, based on similar triangles. You should work out **what equipment** you would need, and **how it would be used** (including any **preparations** that you may need to make).
- Once you have devised your method, try it. Does your method work? How accurate was your result? What problems did you encounter?

Round Up

This Investigation shows one example of where similar triangles occur in nature.
Similar shapes appear elsewhere too — for example, when you make a copy of a picture.

Chapter 5

Data Sets

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Section 5.1 introduction — an exploration into:

Estimation Line-Up

This Exploration is all about jellybeans. Remember it's math though... so actually you have to analyze a list of guesses about the number of jellybeans in a big jar.



You have to estimate **how many** jellybeans there are in a jar.
Write your estimate in large numbers on a **sticky note**.

It might help to stick all the estimates to a wall.

The whole class should arrange their sticky notes **in order** — **from smallest to biggest**.
This Exploration is about using your list of numbers to answer these questions:

- Are the guesses all **quite similar**? Or all **very different**?
- Are the guesses **mostly too high**? Or **mostly too low**?

Example

What is the **range** of this class's estimates?

208 260 270 310 310

Solution

The **RANGE** is the **difference** between the **smallest** estimate and the **largest** estimate.
So **subtract** the smallest estimate from the largest.

For these numbers, the **range = 310 – 208 = 102**.

The **range** tells you **how different** the guesses are. A **large** range means that the guesses are **very different**. A **smaller** range means that they are **more similar**.

Example

What is the **mode** of this class's estimates?

208 260 270 310 310

Solution

The **MODE** is the number that was **guessed most often**.
For these numbers, the **mode = 310**.

You might have **more than one** mode.

The **mode** and **median** are both "**typical values**." They tell you if most guesses were **too high**, or **too low**.

Example

What is the **median** of this class's estimates?

208 260 270 310 310

Solution

The **MEDIAN** is the number in the middle of the **ordered list**.
For the numbers **above**, the **median = 270**.

With an **even number** of estimates, the median is midway between the middle two.
So for the numbers on the **right**, the **median = 265**.

205 208 260 270 310 310

Median is **midway** between **260 and 270**.

Exercises

1. For your class's estimates, find the: **a.** range, **b.** median, **c.** mode
2. Make some statements about your class's estimates. Use the range, mode, and median.

Round Up

These statistics are ways to summarize a whole list of numbers — there's more in this Section.

Lesson 5.1.1

Section 5.1
Median and Mode

The median and mode are both intended to "summarize" a whole data set in a single number. They should show some kind of "most usual" or "middle" value of a set of data. But although they're similar in some ways, they're worked out very differently. This Lesson is about how to find them.

California Standard:

Statistics, Data Analysis, and Probability 1.1

Compute the range, mean, **median**, **and mode of data sets.**

What it means for you:

You'll learn to find the median and mode for sets of data.

Key words:

- central tendency
- data
- median
- mode
- values

Data Sets Can Contain All Sorts of Values

Data sets often contain **numerical values**. For example, the data set below represents the number of hours eight adults said they slept last night.

{12, 8, 7, 8, 8.5, 8, 9, 6} ← Braces "{" and "}" are used to show that values are grouped together in a set.

Data sets can contain **other types** of information too. For example, the data set below represents the hair color of five students.

{blonde, brunette, red, blonde, black} ← If there are two items the same, they're listed twice.

Data sets can contain **huge** amounts of data, and it's very likely that most people won't be interested in reading every single value. So, often a value that represents a **typical value** for the set is used. These typical values are often referred to as **measures of central tendency**.

The Median Is a Measure of Central Tendency

Check it out:

You can only find the median of data sets where you can put the values in order.

The Median

The median of a data set is a value that divides the set into two equal groups — one group containing values bigger than the median, the other containing smaller values.

So the median is the **middle value** when a set of values is **put in order**.

If you have an **odd** number of values, the median is fairly easy to find.

Example 1

Find the median of the following data set: {3, 2, 6, 8, 2, 10, 6, 4, 9}

Solution

First, **arrange the values in order**: {2, 2, 3, 4, 6, 6, 8, 9, 10}

There are nine values, so the median is the **fifth** value.

There are four values less than the median... → {2, 2, 3, 4, **6**, 6, 8, 9, 10} ← ... and four values greater than the median.

median

The median of the data set is 6.

If there's an **even** number of values, then finding the median is slightly trickier because there are **two middle numbers**. Here, you find the value **exactly midway** between the two middle numbers.

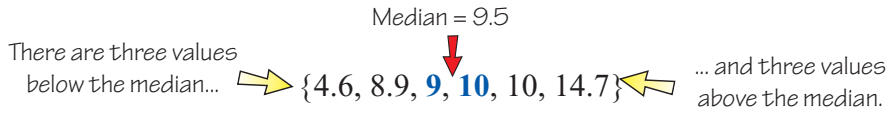
Example 2

Find the median of this data set: {4.6, 8.9, 9, 10, 10, 14.7}

Solution

The values in this data set are already **in order** from least to greatest.

There are six values in the set — the median lies **midway** between the **third and fourth** values.



The third and fourth values are 9 and 10 — so the median is **9.5**.

Guided Practice

Find the median of each of the following data sets.

1. {12, 8, 10, 19, 21, 7, 14}
2. {$101, $201, $150, $198, $300}
3. {5, 8, 3, 6, 12, 9, 5, 5, 4, 11}
4. {–6, –3, 7, 4, –2, –2, 5, 2}
5. {–2.1, 5.7, 8.1, –10.2, –100, 42.778}

Check it out:
It's easy to leave out a value when you're rewriting the data set in order. It's safer to copy the data in the original order first, then cross out the values as you rewrite them.

The Mode Is Another Measure of Central Tendency

The Mode

The mode of a data set is the value that occurs most often.

To find the mode of a data set, look for the value that's listed more than any other value.

Check it out:
The mode is sometimes called the modal value. So in Example 3, the modal value is 2.

Example 3

Find the mode of the data set: {17, 2, 6, 8, 2, 10, 4, 35, 10, 7, 2}

Solution

The number 2 occurs three times — this is more than any other value in the set.

{17, **2**, 6, 8, **2**, 10, 4, 35, 10, 7, **2**}

So the mode is 2.

Check it out:
To find the mode, you don't have to write the values out in order. However, you might still find it useful to do this, as it makes it easier to see which value is repeated most.

Some Data Sets Have No Modes — Others Have Many

Data sets **don't** always have one mode — as these examples show.

Example 4

Find the mode of this set of data:
{brown, blue, green, blue, yellow, brown, orange, white}

Solution

Blue and brown both appear twice. No other color appears more often.
So the data set has two modes, blue and brown.

Check it out:
The mode is a good measure of central tendency to use if your data set contains nonnumerical values.

Example 5

Find the mode of this set of data: {3, 5, 19, 5, 3, 19}

Solution

Each number occurs twice — no value occurs more often than the others.
So this data set has no modes.

Guided Practice

Give the mode(s) of the following data sets.

6. {$12, $8, $7.50, $7.50, $10, $8, $8, $9.50}

7. $\{\frac{2}{5}, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{4}\}$

8. {0, 1, –3, 5, 1, 0, –3, –3, 0}

Independent Practice

Give the median and mode(s) of the data sets in Exercises 1–4.

1. {42, 56, 73, 64, 42}

2. {2, 5, 7, 3, 8, 10, 14}

3. {$16, $28, $20, $15}

4. {0.1, 0.4, 0.7, 0.4, 0.5, 0.7}

5. Write a set of data for which the mode and the median are the same.

Exercises 6–7 are about Rick's survey of car colors.

6. Rick listed the colors of the 25 cars in a parking lot. The mode for his list is blue. What does this tell you?

7. There are only 5 car colors on Rick's list: white, red, black, blue, and green. The mode is blue. What can you say about the possible minimum and maximum number of blue cars? Explain your answer.

In Exercises 7–10, find a number for each blank so the median is 12.

8. {42, 2, 5, 7, 36, __}

9. {4, 12, 18, __}

10. {2, 5, 14, 12, __}

11. {6, 6, 7, 9, 11, 17, 17, 20, 23, __}

Decide whether each statement is true or false. Explain your answers.

12. The median of a data set always equals one of the data values.

13. Not all data sets have a median.

Now try these:
Lesson 5.1.1 additional questions — p451

Round Up

These typical values beginning with "m" can get confusing. The median is the middle number when they're arranged in order, and the mode is the most common value. There's another similar "m" coming up in the next Lesson too — the mean.

Lesson 5.1.2

Mean and Range

California Standard:
Statistics, Data Analysis, and Probability 1.1
Compute the range, mean, median, and mode **of data sets.**

What it means for you:
You'll learn how to find the mean and range for a set of data. Then you'll see how the mode, mean, and median don't all make sense in every situation.

Key words:
- mean
- measure of central tendency
- sum
- range
- mode
- median

Don't forget:
Finding the sum of a set of numbers means adding them all up.

Don't forget:
There's more about rounding in Section 1.4.

Check it out:
The mean is also used to calculate a student's GPA (grade point average).

The mean (like the mode and median) is another way of finding a "typical value" of a data set.

The range, on the other hand, gives a different kind of information — it tells you how spread out the data is.

The Mean Is Also a Measure of Central Tendency

The **mean** is often referred to as the **average**.
You have to do a **calculation** to find it:

> **The Mean**
> The mean of a data set is found by:
> (i) adding up all the items in a data set, and then
> (ii) dividing by the number of items in the data set.

Example 1

Find the mean of the data in the set {2, 7, 4, 8, 10, 24, 57, 39, 8}.

Solution

First, find the **sum** of the numbers in the set:

Sum = 2 + 7 + 4 + 8 + 10 + 24 + 57 + 39 + 8 = **159**

Next **divide** the sum by the number of values in the set.

Mean = 159 ÷ 9 ← *There are 9 values in the set.*

= **17.7**

The exact answer was 17.666666...

(You often have to round mean values to a sensible level of accuracy.)

Guided Practice

Calculate the mean of each data set in Exercises 1–4.
Round your answers to the nearest hundredth.

1. {34, 67, 65, 45, 78, 35, 90}
2. {2, 3, 3, 3, 2, 4}
3. {4.2, 4.5, 4.8, 4.9, 4.5}
4. {12, 12, 24, 23, 26, 32, 14, 18, 18}

5. In a college course, a student's final score is the mean of three tests. To earn a B grade, students need a final score of at least 80. A student earns 78, 91, and 73 points on the three tests. Will this student get a B?

Use the Mean to Find Missing Values

On its own, the mean **doesn't** tell you anything about individual values in a data set. But you can use it to find **one missing value** if you know the rest.

Suppose you've taken five tests, but can't find one result. If you know your **mean score**, you can use it to figure out the missing grade.

Example 2

After taking five tests, Tiffany earns a grade average of 85%.
She can only find four of her test papers. On these, she scored 90%, 82%, 87%, and 98%.

Use the mean to find out her missing test score.

Solution

You have to work backward to solve this sort of problem.

You know that the **mean** is the **total** of all five test scores **divided by** the number of tests (5).
In other words: mean = sum of all five scores ÷ 5
mean = 85

This means you can multiply 85 by 5 to find the sum of the 5 scores.
Sum of all five scores = 85 × 5 = **425**

Now compare this sum to the sum of the four test scores she already knows: 90 + 82 + 87 + 98 = **357**

357 is 68 less than 425.

So the missing test score is 68%.

Check it out:
You can use a variable such as x to stand for the missing score.
$90 + 82 + 87 + 98 + x = 425$
So $357 + x = 425$, which means $x = 425 - 357 = 68$.
See Lesson 2.2.3 for more information about solving this kind of equation.

Guided Practice

6. The mean of two numbers is 76. If one number is 15, what is the second number?

7. Max's final grade is the mean of four tests.
He wants to earn a final grade of 90.
He got 83, 95, and 88 on the first three tests.
What must he get on the fourth test to earn an average of 90?

The Range: Subtract the Least from the Greatest

The range tells you how **spread out** the data is. A set of data that has a **big difference** between the highest and lowest values will have a much bigger range than one in which the values are all **fairly similar**.

The Range
The range of a data set is the difference between the greatest and least values in the set.

Example 3

Find the range of heights of these basketball players:
Player 1: 195 cm; Player 2: 210 cm; Player 3: 202 cm; Player 4: 180 cm

Solution

Subtract the least value from the greatest: 210 cm – 180 cm = **30 cm**

The range of the heights is 30 cm.

Example 4

A data set contains 7 values. The range of these values is 9.
Suggest a data set that meets this description.

Solution

You know that the difference between the highest and lowest values is **9**.

Pick any number for the highest value — 11, say.
Then the lowest value must be 11 – 9 = **2**.

But you can pick any other values for the remaining 5 items, as long as none of them is less than 2 or greater than 11.
So one possible set of values is **{11, 8, 3, 3, 2, 2, 4}**.

There are many different possibilities.

Check it out:

If the highest value is 11 and the lowest is 2, then for the range to stay as 9, the other data values must not be greater than 11 or less than 2.

Check it out:

There are many possible answers to Example 4, since you can pick any value to use as the greatest value.
So another possible answer is {14, 5, 7, 8, 6, 10, 10}.

Guided Practice

Calculate the range of the data sets in Exercises 8–9.

8. {90, 120, 80}

9. {1.2, 1.4, 1.1, 2.5, 3, 2.2}

10. The range of the set of ages of people at a party is 12.
Suggest possible ages for the youngest person and the oldest.

Independent Practice

Calculate the mean and range for each data set in Exercises 1–2.

1. {5, 9, 3, 12}

2. {–5, 7, 18, –1, 6}

For Exercises 3–5, use the data set {19, 13, 7, 2, 1, 1, 25, 4}.
State how many values in the set are:

3. greater than the mean.

4. less than the mean.

5. equal to the mean.

Each set in Exercises 6–9 has a mean of 5. Find the missing values.

6. Set A {1, 9, 7, 8, 2, __}

7. Set B {1, 4, 10, 5, 6, __}

8. Set C {5, 4, 6, 5, 4, __}

9. Set D {10, –2, 0, __}

10. The range of a data set is 12.
If the greatest value is 188, what is the least value?

11. Which one data value could be removed from the data set below to give it a range less than 9? {1, 2, 4, 6, 7, 7, 9, 11}

Now try these:

Lesson 5.1.2 additional questions — p451

Round Up

Now you know about the range and three measures of central tendency — mean, mode, and median. You've also seen a measure of spread — the range. There's more about these next Lesson.

Lesson 5.1.3

Extreme Values

The last two Lessons have looked at the median, mode, and mean. These are "typical values" — values that fall somewhere in the middle of your data. This Lesson is about values that are "right at the end" of your data — these are called extreme values.

Extreme Values Are Sometimes Called Outliers

An **outlier** is a data value that is **much greater than** or **much less than** other values in the data set. For example, the following set of numbers contains an outlier: {9.99, 10.03, 10.08, 10.09, 10.10, 10.12, 10.19, 13.01}. The value 13.01 is a lot higher than all the others.

There are various reasons why some values might lie far from the others.

(i) Sometimes **measurements are made incorrectly** (for example, if someone used the wrong end of a ruler to measure something).

(ii) Sometimes the measurement is "**unfair**" in some way, because of a problem, perhaps. For example, the eight numbers in the set above are times of athletes running a 100-meter race. The first seven athletes finished in times very close to 10 seconds. Unfortunately, the final athlete fell over, tore a muscle, and limped to the end of the race.

(iii) Sometimes the reading is a **genuinely unusual, but correct, value**. For example, some people are much taller than average, while others are much shorter.

Outliers can have a really dramatic effect when you calculate a **mean** value. In cases (i) and (ii) above, it would probably be best to **ignore them** in your calculation. In case (iii) though, you shouldn't really just ignore them — they're as genuine as any other value.

Example 1

This data set shows the times in seconds that it took for a ball to fall through a tube in a science experiment: {2.1, 2.3, 2.5, 2.5, 2.6, 7}
Identify the outlier. Then calculate the mean with and without the outlier, and suggest which would be better to use as a typical value.

Solution

The outlier is **7 seconds**. It's **much** bigger than the others.
The mean with the outlier is:
$(2.1 + 2.3 + 2.5 + 2.5 + 2.6 + 7) \div 6 = 19 \div 3 =$ **3.2 seconds**
The mean without the outlier is:
$(2.1 + 2.3 + 2.5 + 2.5 + 2.6) \div 5 = 12 \div 5 =$ **2.4 seconds**

Here, the measurement of 7 seconds probably happened because there was either: **a problem carrying out the experiment** (maybe the ball got stuck), or a **problem with the measurement** (maybe the stopwatch was started early accidentally).
So the mean of 2.4 s is probably more appropriate.

California Standards:

Statistics, Data Analysis, and Probability 1.1
Compute the range, mean, median, and mode of data sets.

Statistics, Data Analysis, and Probability 1.3
Understand how the inclusion or exclusion of outliers affects these computations.

Statistics, Data Analysis, and Probability 1.4
Know why a specific measure of central tendency (mean, median) provides the most useful information in a given context.

What it means for you:
You'll see how extreme values can affect calculations of the mean, mode, and median, and why this is important.

Key words:
- mean
- measure of central tendency
- sum
- range
- mode
- median

Check it out:
The range would also be heavily affected by an outlier.

Check it out:
Always try to find out the possible reason for an outlier before you ignore it.

Example 2

This data set shows the weights in pounds (lb) of six dogs brought into a veterinary practice: {8, 22, 25, 33, 36, 42}.
(All the weights are genuine.)
Identify the outlier. Then calculate the mean with and without the outlier, and suggest which would be better to use.

Solution

The outlier is **8 lb**. It's **much** smaller than the others.
The mean with the outlier is: 166 ÷ 6 = **27.7 lb**
The mean without the outlier is: 158 ÷ 5 = **31.6 lb**

Here, it's probably best to **include** the outlier. It's a small data value, but it's a genuine weight of one of the dogs.
So the mean of 27.7 lb is probably more appropriate.

Check it out:
The very low weight might be from a different breed, and so perfectly normal.

Guided Practice

1. In which set or sets from A to D might 99 be considered an outlier?
A: {0, –3, 8, 99}
B: {99, 106.4, 99.8, 99.95, 101.6}
C: {88, 100, 99, 104, 89, 91}
D: {413, 99, 526, 480, 475}

Use the following data set for Exercises 2–5: {12, 15, 19, 20, 200}
2. Find the mean of the data set with the outlier included.
3. Find the mean of the data set without the outlier included.

Say which of these answers is appropriate to use if the data shows:
4. the correctly measured lengths of different sea snakes, which are to be used by the makers of snake antivenom.
5. the distances a javelin was thrown by members of a sports club, though the final value was the result of a person typing the result into a computer incorrectly.

Choose the Best Measure for the Situation

You've seen that outliers can have a big effect on the mean. Sometimes, outliers can make the mean **unrepresentative** of the rest of the data set.

Example 3

A doctor sees 15 patients during the morning. The length of time he spent with each patient is recorded below to the nearest minute.
{5, 7, 3, 3, 78, 11, 10, 8, 6, 3, 3, 8, 9, 8, 11}

Find the mean. Suggest a problem with using this as a typical time.

Solution

Mean = Sum of all 15 times ÷ 15 = 173 ÷ 15 = **11.5 minutes**.

The mean is greater than all but one of the values in the data set. The very high value of 78 "pulls" the mean upward, so it is not very typical of the values in the data set.

Check it out:
If a few values are very different from the majority, they'll often cause the mean value to not be representative.

Check it out:
There's more about how the mean is affected by extreme values in Lesson 5.2.2.

This is a problem with using the **mean** as a typical value — it's **heavily affected** by extreme values.

One solution is to use a different measure of central tendency.

Example 4

Using the same data set as in Example 3, find the median and mode. Choose the best measure(s) of central tendency for this data.

Solution

The data set is: {5, 7, 3, 3, 78, 11, 10, 8, 6, 3, 3, 8, 9, 8, 11}

Mode: The number 3 occurs more than any other number, so the **mode = 3 minutes**.

Median: the values written in order are:
3, 3, 3, 3, 5, 6, 7, 8, 8, 8, 9, 10, 11, 11, and 78.
The median is the middle value, so the **median = 8 minutes**.

The mode is equal to the lowest value, and you've already seen that the mean is higher than all but one value, so neither of these is very typical of the data set.
The median is the most representative measure of central tendency.

Guided Practice

For each data set below, find (where possible) the mean, median, and mode. Then say which would be the most suitable measure of central tendency to use, and explain why.
6. Math test scores: {75, 84, 88, 72, 64, 10, 92, 87}.
The outlier was scored by someone new to the town, who had not been to a lesson in this school before.
7. Weekly allowance: {$10, $15, $10, $10, $7, $50}.
This data is to be used to see how many weeks it will take a group of students to raise enough money to throw a birthday party.

Check it out:
If you can't put the data into any sensible kind of order, the mode is the only measure of central tendency that's possible.

Independent Practice

1. In which data set(s) below might –4 be considered an outlier?
W: {0, –6, 2, 12, –3, –8, 11} Y: {–3.8, –4.2, –3.3, –5.1, –2.7}
X: {29.4, 33.7, 39.4, 28.9, 31.1} Z: {3.8, 4.2, 3.3, 5.1, 2.7}

For each data set below, find (where possible) the mean, median, and mode. Then say which would be the most suitable measure of central tendency to use, and explain why.
2. Shoe sizes: {5, 7, 6, 7, 5, 5, 6, 6, 7}
3. Age of party guests: {14, 18, 17, 13, 14, 11, 14, 14, 73}
4. Pets owned: {dog, rabbit, dog, cat, gerbil, dog, cat, dog}

Now try these:
Lesson 5.1.3 additional questions — p452

Round Up

In some cases, you can just ignore outliers — they're sometimes just "mistakes." But you shouldn't just ignore outliers because they're inconvenient — that's "fiddling your data." You can always use the median or the mode if you want a typical value that isn't heavily affected by extreme values.

Lesson 5.1.4

Comparing Data Sets

California Standard:
Statistics, Data Analysis, and Probability 1.1
Compute the range, mean, median, and mode of data sets.

What it means for you:
You'll see how you can use the mean, median, mode, and range to compare data sets.

Key words:
- measure of central tendency
- mean
- mode
- median
- spread
- range

Don't forget:
There's a lot more about how to work out the mean, median, mode, and range in Lessons 5.1.1 and 5.1.2.

Check it out:
The spread of a data set means how spread out (or how close together) all the values are.

You've seen how to find the mean, median, mode, and range of a data set. The next step is to see how you can use them to compare data sets.

Two Key Concepts: "Typical Value" and "Spread"

With any data set, there are **two** basic things you'll really need to know:
1) a "**typical value**,"
2) **how spread out** the values are.

You've seen that you can find a **typical value** in different ways — you can use either the **mean**, **median**, or **mode**.

You've also seen that you can describe **how spread out** your data is, using the **range**.

Example 1

Give a description of the data set below.
{4, 6, 3, 4, 5, 2, 9, 8, 2, 6, 6}

Solution

There are 3 types of typical values — the mean, the median, and the mode.

Mean $= \dfrac{4+6+\ldots+6+6}{11} = \dfrac{55}{11} = 5$

Median... put the data in order first: 2, 2, 3, 4, 4, 5, 6, 6, 6, 8, 9.
There are 11 values, so the median is the 6th value, which is **5**.
Mode = 6

Then you can describe how spread out your data is by looking at the range. Here, the **range** = 9 – 2 = **7**.

Guided Practice

In Exercises 1–5, give a summary of each data set.
1. {8, 4, 7, 4, 5, 4}
2. {21, 11, 22, 14, 17, 13}
3. {–12, –15, –15, –11, –9, –10}
4. {1, 5, 2, 7, 3, 7, 4, 5, 9, 10, 14, 3, 2, 5, 13}
5. {–5, 5, 2, 7, 3, 7, 4, 5, 9, 10, 20, 3, 2, 5, 13}

6. What do you notice about the sets in Exercises 4 and 5?

Compare Data Sets Using Typical Value and Spread

If you have two sets of data, you may want to **compare** them.
For each set, work out a "**typical value**," and **how spread out** the data is.

Example 2

Alfredo and Joel are both on a long jump team.
During the course of the season, they each do a series of jumps.

Alfredo's jumps (in meters): 5.1, 5.42, 6.01, 4.46, 5.33, 5.12, 4.42, 6.02.
Joel's jumps (in meters): 5.23, 5.15, 5.3, 5.33, 5.22, 5.18, 5.3.

Compare their performances over the course of the season.

Solution

Work out the mean, median, mode, and range for both jumpers.

Alfredo first:

Mean $= \dfrac{5.1 + \ldots + 6.02}{8} = \dfrac{41.88}{8} =$ **5.235 meters**

Median... put the data in order first:
4.42, 4.46, 5.1, 5.12, 5.33, 5.42, 6.01, 6.02
So the **median is 5.225 meters**.
All the values are different, so there is **no mode**.

Range = 6.02 – 4.42 = **1.6 meters**.

Now Joel:

Mean $= \dfrac{5.23 + \ldots + 5.3}{7} = \dfrac{36.71}{7} =$ **5.244 meters (to 3 decimal places)**

Median... put the data in order first: 5.15, 5.18, 5.22, 5.23, 5.3, 5.3, 5.33
The **median is 5.23 meters**.
The mode = 5.3 meters.

Range = 5.33 – 5.15 = **0.18 meters**.

You can see that for both jumpers, the mean and the median are very similar. Both have a "typical jump" of about 5.22–5.24 meters.

However, the big difference is in the boys' ranges. Alfredo's range is much greater than Joel's. This means that **Joel is a lot more consistent than Alfredo**, or **Alfredo's jumps are a lot more variable than Joel's**.

Don't forget:
The mean, median, and range are all different kinds of "typical value."
The range is a measure of spread.

Don't forget:
A large range might mean that all the values are very spread out. Or it might mean that there are one or two outliers. Look at the actual values to check.

Guided Practice

Compare the pairs of data sets in Exercises 7–8.

7. {6, 8, 9, 5, 7, 8, 5} and {1, 4, 7, 9, 3, 4}

8. {7, 4, 5, 7, 2, 8} and {28, 34, 33, 29, 33, 34}

9. Sam and Eddy each have a collection of earthworms.
Sam's worms have lengths (in cm) of 5.2, 8.3, 4.4, 6.7, 9.2, and 10.1.
Eddy's worms have lengths (in cm) of 4.6, 6.3, 5.5, 6.6, 6.9, and 7.1.
Compare the two collections.

Example 3

Ryan and Hap scored the following points in 10 games of basketball.
Ryan: 10, 5, 6, 14, 3, 2, 8, 4, 16, 6
Hap: 8, 7, 8, 10, 6, 8, 9, 4, 10, 8

(i) If the team is behind by 8 points, why might the coach put Hap in?
(ii) If the team is behind by 14 points, why might the coach put Ryan in?

Solution

Compare the two sets of data.
Ryan: mean = 7.4; median = 6; mode = 6; range = 14
Hap: mean = 7.8; median = 8; mode = 8; range = 6

(i) Hap has a **median of 8**, meaning he has scored 8 in **at least half** the games he has played. Ryan's **median is 6**, which means he scored 6 points or fewer in half of his games.

(ii) Hap's range is much lower, meaning he is more consistent. But his highest score is only 10, showing that he may not be able to score really highly in a single game. Ryan, however, has a **range of 14**, showing that he has scored **at least 14 points** in a game before.

Don't forget:
A small range shows consistency — with no really big or really small values. A bigger range shows there are some much bigger or some much smaller values.

Guided Practice

One-day first aid courses are taught on Wednesdays and Saturdays. The age of the participants in each course is summarized below.
Wednesday class: mean = 39.75; median = 36; mode = 28; range = 42
Saturday class: mean = 23.25; median = 21.5; mode = 13; range = 30
10. Students in school are unable to attend one of the classes. Using the statistics, which would you say students cannot attend?
11. What are the maximum possible ages of the people in each class?

Independent Practice

Give a summary of each data set in Exercises 1–3.
1. {56, 78, 45, 66, 49, 71}
2. {111, 124, 109, 105, 122, 106, 99, 96}
3. {2.31, 2.43, 2.54, 2.12, 2.54, 2.33, 2.65}

Compare the data sets in Exercises 4–5.
4. {82, 63, 71, 78, 58} and {152, 133, 141, 148, 128}
5. {1.2, 1.7, 1.4, 1.6} and {0.7, 2.2, 0.9, 2.1}

6. A research company reported that the typical teacher in the U.S. is a 43-year-old female. What measure of central tendency might the company have used to determine that the typical teacher was female? Does this mean there are no male teachers? Explain your answers.

Now try these:
Lesson 5.1.4 additional questions — p452

Round Up

The mean, median, mode, and range are used all the time in real life to summarize the results of surveys. It's important you understand what each piece of information tells you.

Lesson 5.2.1

California Standards:

Statistics, Data Analysis, and Probability 1.1

Compute the range, mean, **median, and mode of data sets.**

Statistics, Data Analysis, and Probability 1.2

Understand how additional data added to data sets may affect these computations.

What it means for you:

You'll see how the mode, median, and range of a data set change if extra values are included.

Key words:

- data set
- mode
- median
- range

Check it out:

If you had a huge data set, containing hundreds and hundreds of values, then it would be far less likely that an extra three responses would change the mode.

Don't forget:

The mode is sometimes called the modal value.

Section 5.2
Including Additional Data: Mode, Median, and Range

The mode, median, mean, and range of a data set can change as new data is added. Adding just one extra number to the data set might (or might not) be enough to change any of them. This Lesson looks at how including new data can affect the mode, median, and range.

Adding Data Values Can Change the Mode

The **mode** of a data set is the item that **occurs most often**.
New data can cause the most common value to change.

Example 1

Joe has conducted a survey asking the number of TVs in people's households. The data set is shown below.
{0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 7}

The next day, three more completed surveys arrive in the mail.
The new responses are {1, 3, 3}.

Find the mode of the data set:
(i) not including the extra responses,
(ii) including the extra responses.
What effect do the extra responses have on the mode?

Solution

(i) The mode of the original data set is 2 (there are eight values of 2).
(ii) The mode of the new data set is 3 (there are now nine values of 3, but still only eight of 2).

Here, the extra data **increased the mode from 2 to 3**.

If the three extra responses in Example 1 had been different, the mode **may not** have been affected.

For example, the responses {0, 2, 5} would **not** have changed the mode.

Guided Practice

1. If 9 is added to the data set {1, 5, 5, 5, 7, 9, 9, 10, 15}, how is the mode affected?

The number of actors needed for six short plays are 4, 6, 4, 5, 6, and 6.
2. What is the modal (most common) number of actors?
3. Three more plays are added to the list. The numbers of actors needed are 4, 4, and 5. What is the modal number of actors now?
4. The data set is now {4, 6, 4, 5, 6, 6, 4, 4, 5}.
Could a single number be added that would change the mode?
Explain your answer.

Check it out:
If there are an odd number of values in the data set, the median is the actual value in the middle of the ordered data set.
If there are an even number of values, then the median is midway between the middle two values.
See Lesson 5.1.1 for more information.

Adding Data Values Can Also Change the Median

The **median** is the **middle value** when all the values of a data set are arranged in order.

Adding a new value can easily **change** the median.

Example 2

The heights of four friends are 156 cm, 157 cm, 164 cm, and 168 cm. Another friend joins them. His height is 207 cm.

How does adding this friend's height affect the median?

Solution

First, you need to find the original median.
So arrange the original data set **in order**.

Original data set: {156, 157, 164, 168}

Old median = 160.5 cm

The original median is halfway between 157 cm and 164 cm (the 2nd and 3rd values).

Check it out:
The extra height is greater than the old median, so the new median is greater than the old one.

Now insert the additional data value into the set, keeping the heights in order. Next, find the **new median** and compare it to the original.

new value

New data set: {156, 157, 164, 168, **207**}

New median = 164 cm

The median is now the 3rd value, 164 cm.

Check it out:
Notice how the increase in the median is only a few centimeters, even though the new friend is much taller than the other friends.

The median increased from 160.5 cm to 164 cm.

Example 3

Another friend joins the friends from Example 2. Her height is 146 cm. How is the median affected now?

Solution

Now the ordered data set is: {146, 156, 157, 164, 166, 168}.

There are 6 values, so the new median is midway between the 3rd and 4th values. **The new median is 160.5 cm.**

Check it out:
This time, the new height is less than the previous median of 164 cm, so the median has fallen. Actually, the median has returned to its original value of 160.5 cm. This is because the two new heights were added to "either end" of the data set, meaning the middle value has remained unchanged.

Guided Practice

5. What is the median of the data set {8, 3, 4, 9, 12, 14}?
6. If a value of 1 is added to the data set in Exercise 5, how is the median affected?
7. Choose another number that could have been added for the median to have changed as you described in your answer to Exercise 6?
8. You add two new values to a data set — one less than the old median and one greater than the old median. What happens to the median?

Check it out:
Notice that adding new data points can only increase the range — never decrease it. The only way to decrease the range would be to remove the greatest or least data point.

New Data Values Can Increase the Range

The **range** is the difference between the greatest and least values.
If a new data point is either **bigger than the greatest value** or **smaller than the least value**, then the range will increase.

Example 4

Would the range of the data set {2, –5, 4, 9, 10, 12} change if:
(i) the value 15 were added?
(ii) the value 7 were added?

Solution

The current range is 12 – (–5) = 17.
(i) 15 is greater than the current greatest value, so **the range will increase**. In fact, the new range will be 15 – (–5) = 20.
(ii) 7 is **not** greater than the current greatest value or less than the current least value, so **the range will not change**.

Check it out:
You can add as many values as you like without changing the range as long as the values are no less than the original least value and no greater than the original greatest value.

Example 5

In a race, the first runner completes the course in 3 hours 10 minutes. The cut-off time, after which no further times are recorded, is 5 hours. Several racers finish at 5 hours and have their times recorded.

Find the range of recorded race times.

How would the range be affected if there were no cut-off time and other, slower, racers had their finish times recorded?

Solution

The range is: 5 hrs – 3 hrs 10 mins = **1 hr 50 mins**

If there were no cut-off time, and the slowest recorded time was greater than 5 hours, then **the range would increase**.

The range can also be affected if data sets are **combined**.

Example 6

The results of a shorter race for under 20s are shown below.

Age group	Race times (minutes)			Range
Age 10–14	45	43	50	50 – 43 = 7 minutes
Age 15–19	39	46	—	46 – 39 = 7 minutes

As there were so few runners, race officials decide to combine the runners into one category for the awards. How does the range change?

Solution

The new data set is: {45, 43, 50, 39, 46}
So the new range = 50 – 39 = **11 minutes**.

Check it out:
In Example 6, even though both original ranges were 7 minutes, the range increased to 11 minutes when the data was combined.

Guided Practice

For each set in Exercises 9–12, decide whether adding the value 5 to the set will change the range. If the range does change, explain how.

9. Set A {0, 2, 2, 7}
10. Set B {13, 7, 9}
11. Set C {–6, 2, 5, 3}
12. Set D {11, 9, 14, 18, 9}

13. One data value is added to the set {48, 51, 68, 73} and it doubles the range. Name the two possible values that were added.

Independent Practice

For each set in Exercises 1–4, decide whether adding the value 16 will change the mode. If the mode does change, explain how.

1. Set A {4, 10, 16, 11}
2. Set B {400, 30, 30, 16}
3. Set C {6.3, 1.4, 15.6, 12.7, 12}
4. Set D {7, 9, 7, 7, 16}

In Exercises 5–8, use the data set {15, 17, 17, 15, 15, 19}.

5. Find the mode of the data set.
6. Name a data value that could be added to the set so that the mode remains the same.
7. How many data values must be added to change the mode to 17? Name the values.
8. What is the least number of data values that must be added to change the mode to 19? Name the values.

Determine whether the statements in Exercises 9–12 are true or false.

9. Adding a data value to a set of data will always change the mode.
10. For the mode to remain the same, you must add a data value that is equal to the mode.
11. Adding a value to a set may cause that set to have an extra mode.
12. Adding a value bigger than the greatest value of a data set will change the range.

Now try these:
Lesson 5.2.1 additional questions — p452

The ages of children on two different trips to the beach are listed below.
Trip 1 ages: {7, 7, 8, 10, 10, 10, 11}
Trip 2 ages: {7, 7, 9, 11, 12, 12, 12, 13, 13}

13. Find the mode and range for each group.
14. If the two trip groups were combined, what would the new mode and range be?

Don't forget:
Values in data sets don't always have to be integers.

Use the data set {6, 7, 19} for Exercises 15–16.

15. Add the value 1 to the set. How does this change the median?
16. Add the value 7 to the original set. How does the median change?

17. Give an example of a data set with a range of 15 that changes to 21 when three data values are added. Show the original set and the new set that includes three added values.

Round Up

Adding numbers to data sets might (or might not) change the mode, median, or range — it depends on what you add. It's safest to find the measures from scratch using the new data set.

Lesson 5.2.2

Including Additional Data: The Mean

California Standards:

Statistics, Data Analysis, and Probability 1.1

Compute the range, **mean**, median, and mode **of data sets**.

Statistics, Data Analysis, and Probability 1.2

Understand how additional data added to data sets may affect these computations.

What it means for you:

You'll see how the mean of a data set changes if extra data is included.

Key words:

- data set
- mean

Don't forget:

The mean is often called the "average."

Last Lesson you saw how the mode, median, and range of a data set changed if new values were added. The next thing to think about is what happens to the mean if you add in extra values to your data set.

Adding Data Points Can Change the Mean

Remember... you find the mean by **adding** all the values of a data set together, and **dividing** this total by the number of values you have.

Adding **extra values** to a data set usually affects the mean.

Example 1

The heights of four friends are 156 cm, 157 cm, 164 cm, and 168 cm. What is their mean height?

Another friend who is 158 cm tall joins them.
How does his height affect the mean height?

Solution

First calculate the original mean:
Original mean = (156 + 157 + 164 + 168) ÷ 4 = 645 ÷ 4 = **161.25 cm**

Now include the extra height of 158 cm, and calculate the new mean.
New mean = (645 + 158) ÷ 5 = 803 ÷ 5 = **160.6 cm**

You can add the new value to the original sum.

Remember to divide by 5, since there are now 5 heights in the data set.

Adding the new height decreased the mean.
This is because the extra height was less than the old mean.

New data values will usually affect the mean, but **not** always.

Example 2

Find the mean of the data set {8, 17, 36, 15}.
How is the mean affected if an extra value of 19 is added?

Solution

The mean of the data set is: $\frac{8+17+36+15}{4} = \frac{76}{4} = 19$

Now add the new value of 19 and find the new mean.

This is: $\frac{76+19}{5} = 19$

This time, **adding the extra value didn't affect the mean** because the new value was equal to the old mean.

When you **combine** data sets, find the new mean **carefully**.

Example 3

Find the mean of the data set {5, 4, 18, 25}, and the mean of the data set {24, 12}.

What is the mean of the combined data set {5, 4, 18, 25, 24, 12}?

Solution

The mean of {5, 4, 18, 25} is: $\frac{5+4+18+25}{4} = \frac{52}{4} = 13$

The mean of {24, 12} is: $\frac{24+12}{2} = \frac{36}{2} = 18$

Now combine the data sets and find the new mean.

This is: $\frac{5+4+18+25+24+12}{6} = \frac{88}{6} = 14.67$ (to 2 decimal places)

Don't forget:

You could find the new mean by adding the old totals, and then dividing by 6:
(52 + 36) ÷ 6 = 88 ÷ 6
= 14.67

Check it out:

Calculate the new mean properly — by adding up all the values, and dividing by how many there are.
Don't assume the new mean is midway between the two old means.

Guided Practice

The weights of four friends are 132 lb, 155 lb, 147 lb, and 151 lb.

1. Find the mean weight of the four friends.

2. Find the new mean if a friend weighing 140 lb joins the group.

A data set has a mean of 109.2. A new data value is then added. State whether the mean of the data set would increase or decrease if the new data value is:

3. 119 **4.** 96.7 **5.** 115 **6.** 107

7. Find the mean of the data set {125, 165, 133, 129, 100}.

8. This data set is combined with the set {43, 44, 27}. Find the mean of the combined set.

Different Values Affect the Mean to Different Extents

Adding data points **near** the original mean won't change the mean too much. But adding values that are **further away** has more effect.

Example 4

How is the mean of the data set {12, 12, 15} affected if you include
i) the number 14?
ii) the number 40?

Solution

The mean of the original data set is (12 + 12 + 15) ÷ 3 = 39 ÷ 3 = **13**

i) The new mean = (39 + 14) ÷ 4 = 53 ÷ 4 = **13.25**
The mean isn't changed much, since 14 is close to the old mean.

ii) This time, the new mean = (39 + 40) ÷ 4 = 79 ÷ 4 = **19.75**
The mean is quite different this time. This is because the extra value is so far away from the old mean (it's an outlier).

Check it out:

Notice that when 40 is added, the new mean becomes higher than any of the data set's original values.
The mean is no longer very representative of the data set.

Don't forget:

The extra value 40 is an outlier for this data set.
Outliers can have a drastic effect on the mean (see Lesson 5.1.3).

Means Found Using Many Values Are Affected Less

Something else also affects how much a new value will change a set's mean — the **number of values** used to work out the mean originally.

Example 5

Find the mean of each of the following data sets. How is the mean of these sets affected if an extra value of 13 is added to each one?
(i) {7, 9, 13, 11},
(ii) {7, 9, 12, 8, 9, 13, 12, 10, 8, 12, 6, 14, 11, 9}.

Solution

(i) The mean of the first set is: $\frac{7+9+13+11}{4} = \frac{40}{4} = 10$

If the value 13 is added, the new mean is: $\frac{53}{5} = 10.6$

So adding the new value **increases the mean by 0.6**.

(ii) The mean of the second set is: $\frac{7+9+...+11+9}{14} = \frac{140}{4} = 10$

If the value 13 is added, the new mean is: $\frac{153}{15} = 10.2$

So this time, adding the new value **increases the mean by just 0.2**.

Check it out:
The more values that are used to calculate a mean, the less it will be affected by a single new value.

Guided Practice

Use the data set {6, 7, 19} for Exercises 9–10.
9. Find the mean of the original set.
10. Adding which value would affect the mean most: 7, 10, or 11?

11. Leonora asked 3 people how many pets they own. The mean was 2. Esteban did a similar survey of number of pets owned by 40 people, and also found that the mean number was 2. Whose result would be affected more if a person owning 18 pets was included in their data?

Independent Practice

The masses in grams of eight mice were recorded as 22, 25, 24, 22, 26, 25, 21, and 25.
1. What is the mean mass of the mice?
2. A ninth mouse, with a mass of 45 grams, was included in the data set. What is the new mean?
3. A researcher finds data for 100,000 mice from all over the country. The mean was 23 grams. How would including the 45-gram mouse affect this mean?

Now try these:
Lesson 5.2.2 additional questions — p453

Round Up

Adding values to data sets will usually change the mean. The more extreme the new value is, and the fewer values used to find the mean in the first place, the greater the effect.

Lesson 5.3.1

Section 5.3 Analyzing Graphs

When you have lots of data in a long list, it's not always easy to figure out what it means. That's why it's often converted into a graph or chart. These can show you what the spread of the data is like, and which value is most common.

California Standards:

Statistics, Data Analysis, and Probability 1.1
Compute the range, mean, median, **and mode of data sets.**

Statistics, Data Analysis, and Probability 2.3
Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.

What it means for you:

You'll meet some useful types of graphs. You'll see how to take information from those graphs, including finding the mode and range of the data set shown.

Key words:

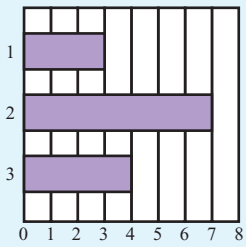
- data set
- frequency table
- bar graph
- line plot
- pictograph
- mode
- range

Check it out:

Frequency tables are sometimes called tally charts.

Check it out:

Bar graphs are sometimes drawn the other way around, with bars going across, and the scale along the bottom.



Data Can Be Recorded on a Frequency Table

Frequency is the number of times something happens.
A **frequency table** is a way to record and organize **data**.

This frequency table shows the results of a survey.
A group of students was asked how many movies they each saw last month.

Number of movies seen	Tally	Frequency
None	II	2
One	𝍸 IIII	9
Two	IIII	4
Three	I	1
Four	𝍸	5
Five		0
Six	I	1

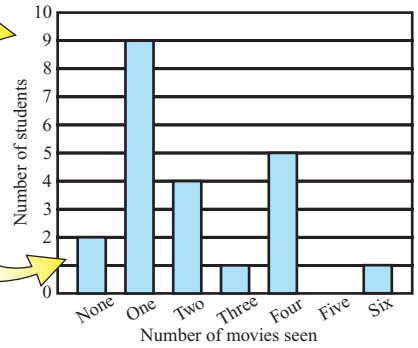
As each student is questioned, one **tally mark** is added in the row that matches their answer.

The **total number** of students who gave each answer is added up when the survey is complete. This total is the **frequency**.

You can use **graphs** and **charts** to show frequency data.

On a **bar graph**, each bar has a **label** which tells you what group that bar **represents**. The **size** of a bar tells you how **large** that group is.

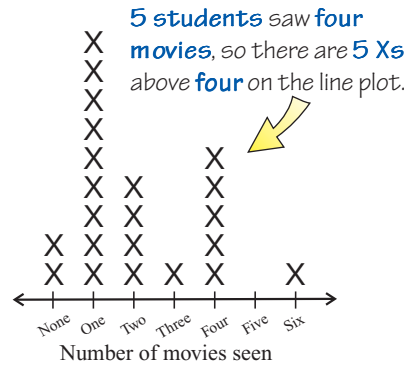
The **scale** shows that the **height** of the bars represents the **number of students**.



The height of the bar labeled "**None**" is **2**.
So **2 students** saw **no movies** last month.

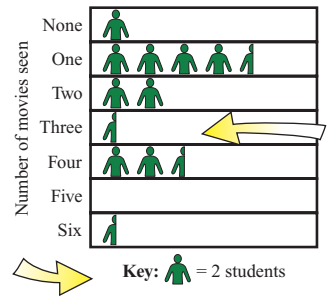
A **line plot** uses **crosses** marked above a **number line** to represent the data.

For any number on the number line, each X above it represents **one student** in the survey who saw that number of movies.



A **pictograph** uses pictures or **symbols** to show the data.

On a pictograph, each symbol often has a value of more than one unit. You need to look at the **key** before you can read the graph.



Guided Practice

Check it out:

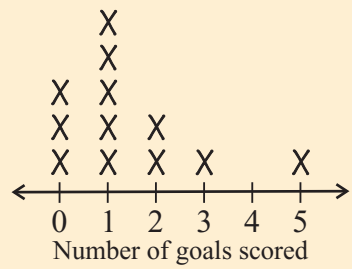
In Exercises 1–3, remember... the number of goals scored is the number underneath the number line. The amount of crosses above each number represents the number of matches in which that many goals were scored.

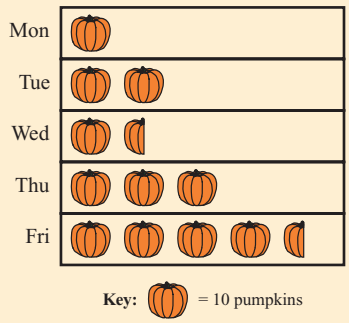
This line plot shows the number of goals scored in the soccer matches played by a school team this year.

1. In how many matches did the team score 0 goals?

2. In how many matches did the team score 2 or more goals?

3. How many matches did the team play this year?





This pictograph shows the number of pumpkins sold by a grocery store during five days in November.

4. When were the most pumpkins sold?

5. How many pumpkins were sold on Wednesday?

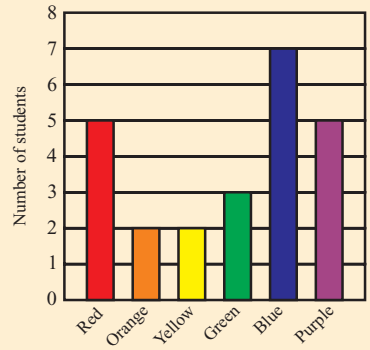
6. On which day were exactly 30 pumpkins sold?

In a survey, some students were asked which of six colors they like best. This bar graph shows the results.

7. Which color was the favorite of 3 students?

8. Which two colors were the least popular?

9. How many students said their favorite color was either red, yellow, or blue?



A Graph Can Tell You the Range and Mode

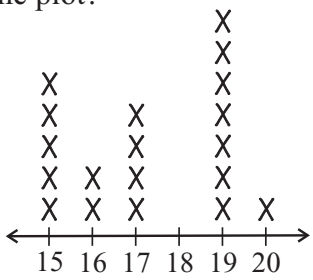
Don't forget:
For a reminder about the mode of a data set, see Lesson 5.1.1. For the range, see Lesson 5.1.2.

The **mode** of a data set is the value that occurs **most often**.

So if a data set is made into a line plot, bar graph, or pictograph, the mode is the value with the **most crosses**, the **biggest bar**, or the **most symbols**.

Example 1

What is the mode of the data set shown on this line plot?



Solution

The mode is the **value** on the line plot that has the **greatest number of Xs** above it.

The number 19 has the most Xs above it.
So the **mode is 19**.

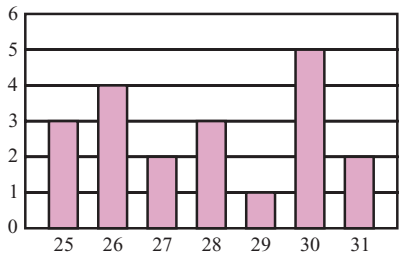
The **range** of a data set is the **difference** between the **greatest** and **least values** in the set.

So it's easier to find the **range** from a graph than from the raw data set because you can clearly see the least and greatest values.

Don't forget:
To work out the range, you need to look at the highest and lowest of the actual values — not how often they occurred.

Example 2

What is the range of the data set shown on this bar graph?



Solution

The different values in the data set are the **labels** on the bars of the graph.

So the least value in the data set is **25**, and the greatest is **31**.

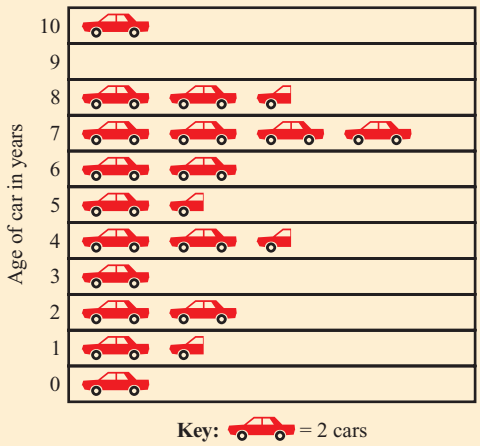
31 – 25 = 6, so **the range = 6**.

Guided Practice

This pictograph shows the ages of the cars parked in a parking lot, to the nearest year.

10. What is the mode of the ages of the cars?

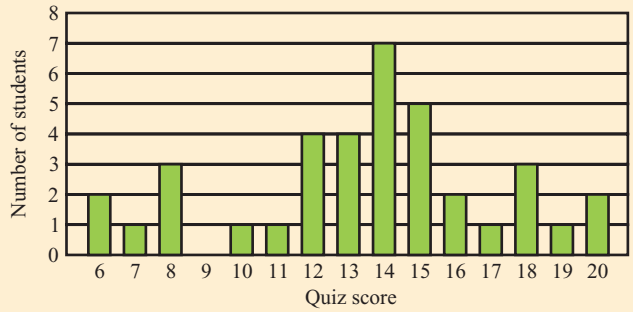
11. Find the range of the ages of the cars.



Don't forget:

Make sure you read from the correct scale. For example, Exercises 12 and 13 are asking for actual scores, not the number of people who achieved that score.

The graph below shows the points out of 20 that the students in one class scored on a math quiz.

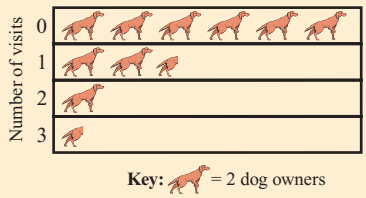


12. What was the mode for the quiz scores?

13. What is the range of quiz scores?

Independent Practice

A number of dog owners were asked how many times they had taken their dog to the veterinarian in the past month.
The results are shown on the pictograph below.



1. How many dog owners took their dog to the vet exactly once?

2. How many dog owners were interviewed in total?

3. What is the mode of the data?

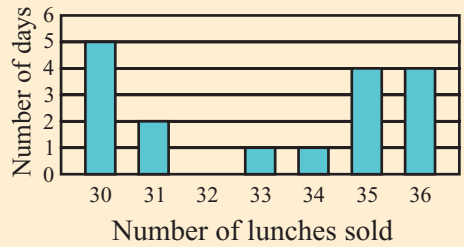
4. What is the range?

The bar graph on the right shows the number of lunches sold each day by a restaurant.

5. Find the mode of the data.

6. Find the range of the data.

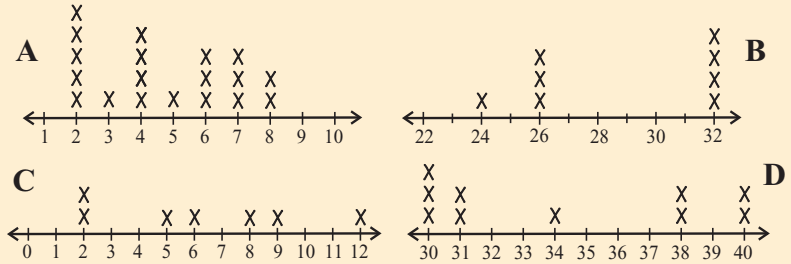
7. For how many days was this data collected?



Which of the line plots below have:

8. a range of 8?

9. a mode of 2?



10. Explain the advantages of looking at data on a line plot instead of in a data set.

Now try these:

Lesson 5.3.1 additional questions — p453

Round Up

You should be able to see how useful these graphs can be. They make it much easier to spot the mode and find the range of a data set. Not only that, they organize the data. As you'll see in the next Lesson, that can help you to find the mean and the median of the data set too.

Lesson 5.3.2

Finding the Mean and Median from Graphs

California Standards:

Statistics, Data Analysis, and Probability 1.1

Compute the range, **mean, median,** and mode **of data sets.**

Statistics, Data Analysis, and Probability 2.3

Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.

What it means for you:

You'll see how bar graphs, line plots, and pictographs can make it easier for you to find the mean and median of data sets.

Key words:

- data set
- bar graph
- line plot
- pictograph
- mean
- median

Don't forget:

For more about the median of a data set, see Lesson 5.1.1.

In the last Lesson, you saw how to find the mode and the range of a data set from a line plot, bar graph, or pictograph.

That's not all you can do with these graphs. You can use them to find the mean and the median too.

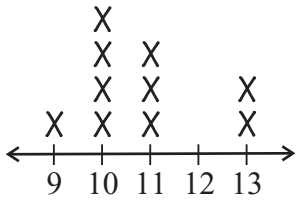
You Can Use a Graph to Find the Median

The median is the **middle value** when a set of values is **put in order.**

A graph can give you the information you need to find the **median** of a data set.

Example 1

What is the median of the data set shown on this line plot?



Solution

Remember that **each X** on the line plot represents **one item** in the data set.

So you can turn the plot back into a set of data:

{9, 10, 10, 10, 10, 11, 11, 11, 13, 13}

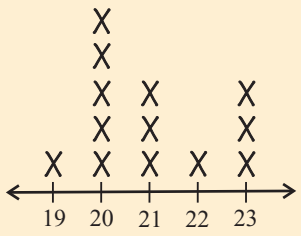
There are **ten values** in the set, so the median is **midway** between the **fifth** and **sixth** values.

The fifth and sixth values are **10** and **11**, so the median is **10.5**.

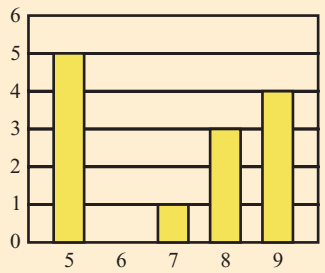
Guided Practice

In Exercises 1–4, find the median of the data set shown on each graph.

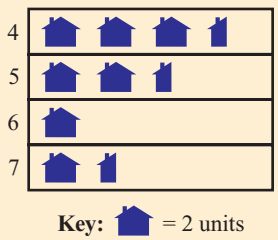
1.



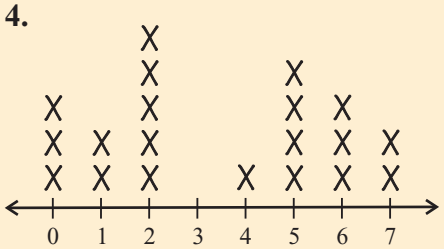
2.



3.



4.



You Can Use a Graph to Find the Mean

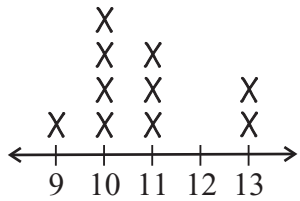
The **mean** of a data set is the **sum** of all the values, divided by the **number of items** in the data set.

There are a couple of ways that you can use a graph to find the mean of a data set.

Don't forget:
See Lesson 5.1.2 for more about the mean of a data set.

Example 2

What is the mean of the data set shown on this line plot?



Solution

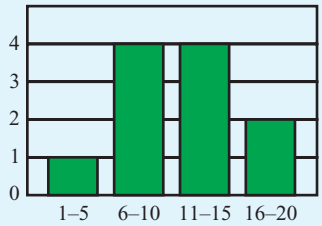
This is the same graph as in Example 1. So the data set is:

{9, 10, 10, 10, 10, 11, 11, 11, 13, 13}

The mean is the **sum** of these values, divided by the **number of values**.

$(9 + 10 + 10 + 10 + 10 + 11 + 11 + 11 + 13 + 13) \div 10$
$= 108 \div 10$
$= \mathbf{10.8}$

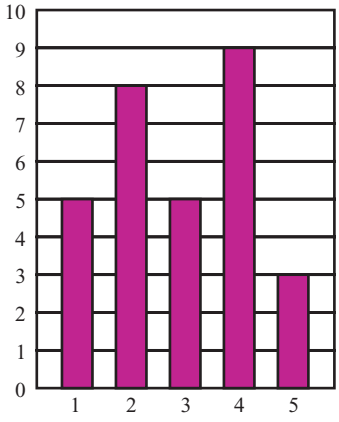
Check it out:
Some graphs organize the data into intervals of values. So a bar graph might have bars representing 1–5, 6–10, 11–15, and so on.



You can't tell from these graphs exactly what the values in the original data set were, so you can't use them to find the exact median or mean.

Example 3

What is the mean of the data set shown on this bar graph?



Solution

The **scale** tells you that there are **5 items** of data with a value of **1**, **8 items** with a value of **2**, and so on.

You can turn this into a **multiplication** to find the **sum** of all the values in the data set.

The data set has:

5 items with the value 1	5×1	$= 5$
8 items with the value 2	8×2	$= 16$
5 items with the value 3	5×3	$= 15$
9 items with the value 4	9×4	$= 36$
3 items with the value 5	3×5	$= 15$
30		**87**

The sum of the values is 87, and there are 30 values.

So the mean is $87 \div 30 = \mathbf{2.9}$.

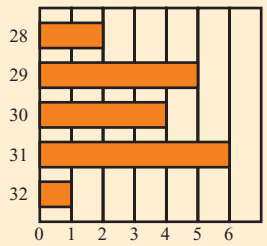
Check it out:
Finding the mean by multiplication is often much quicker than writing out the data set, especially if there are a lot of items in the data set. And it's easy to make a mistake when you have to write out a long list of values.

Guided Practice

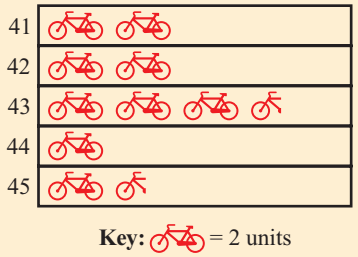
5. Find the mean of each of the data sets shown on the graphs in Exercises 1–4.

In Exercises 6–7, find the mean of the data set shown on each graph.

6.



7.

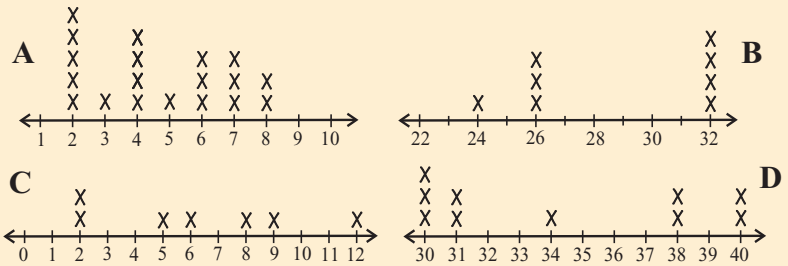


Independent Practice

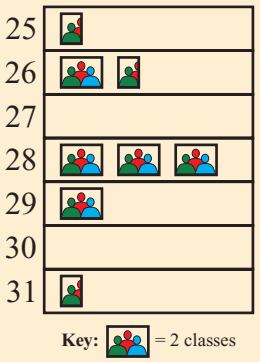
Which of the line plots below have:

1. a median of 32.5?

2. a mean less than 12?



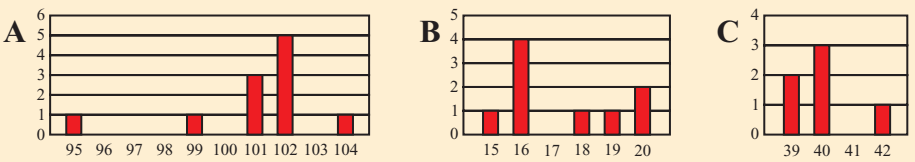
All the classes in an elementary school have from 25 to 31 students.
This pictograph shows the class sizes.



3. How many classes are there in this school?

4. Find the median and mean number of students in each class.

5. Which one of the graphs below represents a data set that has the same mean, median, and mode?



Now try these:

Lesson 5.3.2 additional questions — p454

Round Up

The types of graphs you've used to find the mean, median, mode, and range of a data set are all similar types of graphs. Bar graphs, line plots, and pictographs all show the same sort of information.
There are other types of graphs that can show data in different ways.
You'll see a couple of these in the next Lesson.

Lesson 5.3.3

Other Types of Graphs

You've now seen how to get information from line plots, bar graphs, and pictographs. In this Lesson, you'll learn about a couple of other types of graphs that can show information in different ways.

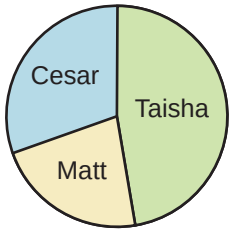
Circle Graphs Show Proportions

Circle graphs divide data into different groups or categories.
The **whole circle** represents **all the data** in a data set.

Circle graphs can show how different groups **compare** to each other, but they don't always give detailed **numerical information**.

Example 1

Three students ran in the election for class president. This circle graph shows what proportion of the votes each candidate received.



List the three candidates in order of the number of votes they received, from most to least.

Solution

The circle graph doesn't tell you **how many** votes, or even what percent the students got. You can still tell that **Taisha** got the **most votes**, because her part of the circle is the **largest**.
Matt's part of the circle is **smallest**, so he got the **least votes**.

The result of the election was:
1st — Taisha
2nd — Cesar
3rd — Matt

Guided Practice

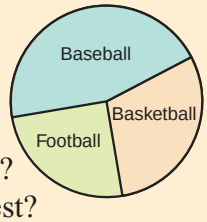
Class A



The students in a 6th grade class were asked which of three sports they like best.
This circle graph shows the results of the survey.

1. Which sport was most popular with Class A?
2. Which sport was least popular with Class A?

Class B



The same number of students in another 6th grade class were asked the same question.
This circle graph was drawn using their answers.

3. Which class has more students who like baseball best?
4. Which class has more students who like basketball best?
5. Which class has more students who like football best?

California Standards:

Statistics, Data Analysis, and Probability 2.3

Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.

What it means for you:

You'll see some of the information you can get from circle graphs and line graphs, and how these types of graphs can help you to compare two data sets.

Key words:

- data set
- circle graph
- line graph
- compare

Don't forget:

See Lesson 3.5.4 for more about circle graphs.

Check it out:

If you didn't know that the graphs represented the same number of students, you couldn't do Exercises 3–5, as circle graphs show proportions. However, because the graphs for Class A and Class B both represent the same number of students, each slice represents actual numbers of students too. Think about this... 10% of 1000 students would be quite a small slice of a circle graph, but represents 100 students. But 50% of 20 students would be half a circle graph, and yet only represents 10 students.

Check it out:

Usually, one of the quantities on a line graph is time. So the line graph would show how some quantity changes as time passes.

Check it out:

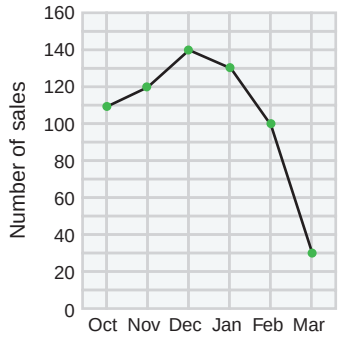
Don't mix up line plots and line graphs. The names are similar, but the two types of graphs are very different.

A Line Graph Can Show Changes Over Time

Line graphs show how **two measures** are **related**.
They're often used to show how a measure **changes** as **time** passes.

Example 2

This line graph shows the monthly sales of winter coats from one store over a period of six months.



What happened to the sales of winter coats from October to December?

What happened to the sales of winter coats from December to March?

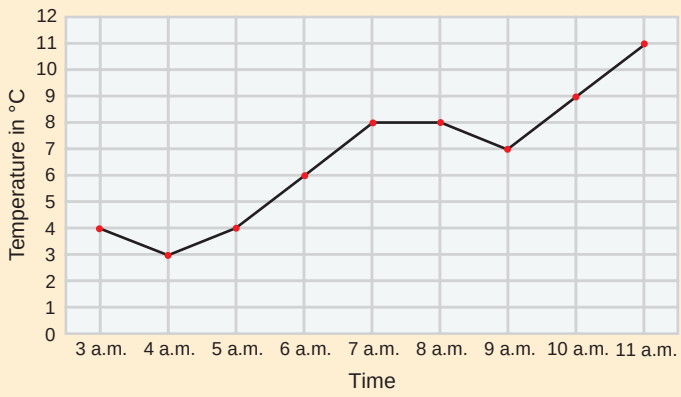
Solution

In October, 110 coats were sold. In December, 140 coats were sold. The sales of coats **increased** from October to December. You can tell this from the graph, because the line is **sloping upward** as you look from left to right.

From December to March, the line on the graph **slopes downward**, because the sales of winter coats **decreased** during this period from 140 in December to 30 in March.

Guided Practice

The line graph below shows the temperature recorded by a weather station at several points during the same morning.



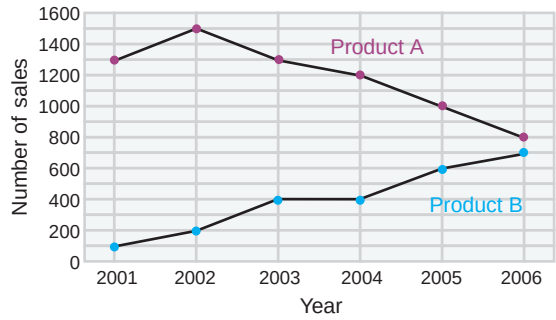
6. What was the temperature at 10 a.m.?
7. At what time was the temperature 6 °C?
8. During which hours did the temperature fall?
9. What was the change in temperature between 7 a.m. and 8 a.m.?
10. Describe the overall change in temperature during the time period shown on the graph.

You Can Compare Two Data Sets on One Graph

With a **line graph**, you can show information about **two data sets** at once. This makes the data easier to **compare**.

Example 3

The line graph below shows the annual sales in one store of two different products over a period of 6 years.
Describe the information shown on the graph.



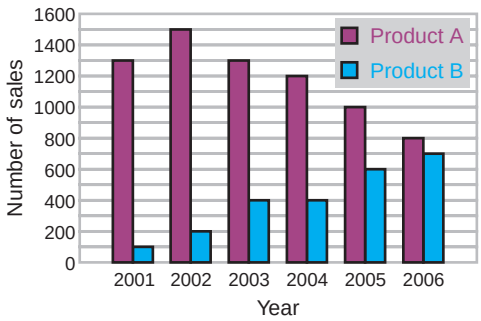
Solution

At the start of the period, the sales of product A were **much higher** than the sales of product B. Over the 6 years, sales of product A **decreased** and sales of product B **increased**. By 2006, the sales figures for the two products were very **similar**.

If the **pattern** of sales shown here **continued**, you would expect sales of product B to **overtake** sales of product A in the next few years.

You can show information about two data sets on other types of graphs. This **double bar chart** uses the same data as the line graph in Example 3.

You could put the same information on a **line plot** or **pictograph**, using different colors or symbols for the different products.



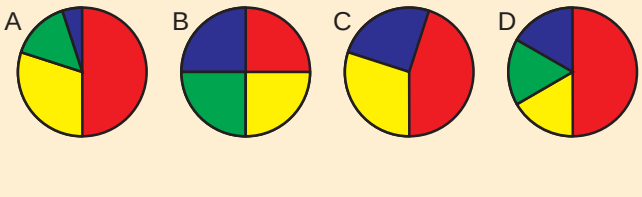
Guided Practice

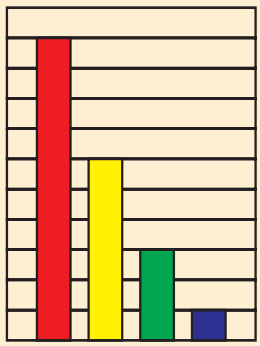
Use the graph shown in Example 3 to answer Exercises 11–14.

11. What was the difference in sales between product A and product B in 2005?
12. Which year had the greatest difference in sales between the two products?
13. In which years were sales of product B double what they had been the year before?
14. In which year were sales of product B the same as they had been the year before?

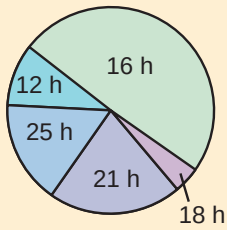
Independent Practice

1. Which of the circle graphs below shows the same data set as the bar graph to the right? Explain your answer.



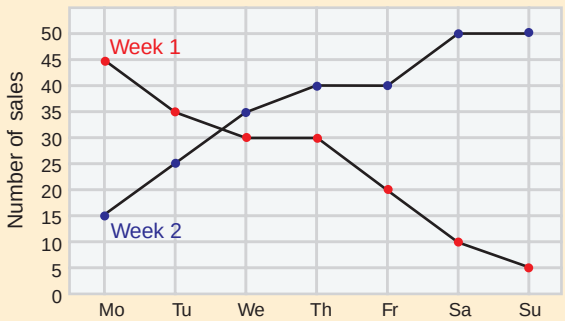


This circle graph shows results of a survey recording the number of hours worked in one week by a group of volunteers.



2. How many people worked for a total of 17 hours?
3. What is the lowest number of hours worked?
4. What is the highest number of hours worked?
5. What is the range of the data set?
6. What is the mode of the data set?

This graph shows how many customers bought frozen yogurt from a particular store during two different weeks.



7. How many people bought frozen yogurt on Thursday in week 1?
8. How many customers bought frozen yogurt on Friday in week 2?
9. How many customers in total bought frozen yogurt during the two weeks?
10. What was the range of sales in week 1?
11. What was the range of sales in week 2?
12. In one of the two weeks, the weather started out cool, but got warmer by the end of the week. Which week do you think that was? Explain your answer.

Now try these:
Lesson 5.3.3 additional questions — p454

Round Up

You should now have an idea of the kind of information you can read or display using different types of graphs. Later in this Chapter, you'll see how the way a graph is drawn can affect what people think when they see it.

Section 5.4 introduction — an exploration into:

Sampling Survey

You need to find out something about the students in your school. Think of ways you could try and find out this information. You could try and ask everybody, but that might take a very long time. Another way is to ask just a small selection of people — a sample.

Work in groups of **four** students.

Your group must first decide what to **find out**. You must then write a **survey question**, such as:

- "How many pets do you have?"

or

- "How many minutes does it take you to get to school?"

Do NOT ask EVERYONE in the class (or the school) the survey question.

Each person in the group will only ask a **selected group of students** — this is called **sampling**.

Each person in your group must use a **different sampling method**. Choose from the methods below:

1. **One person** should ask for **volunteers** to participate in the survey.
 That person must ask the survey question to the **first eight** students who volunteer.

2. **One person** should ask the survey question to the **eight** students sitting **closest to your group**.
 Do not ask members of your group.

3. **One person** should ask the survey question to **the person in each of the other groups whose first name comes last alphabetically**.

4. **One person** should write the names of **every** student in the class on a piece of paper.
 That person must put all the names in a **container**.
 They must then **choose eight names without looking**.
 They must then ask the question to the **eight students whose names they chose**.

Exercises

1. Were the results of each survey method the same? If not, describe the differences.
2. Which of the four methods best represents the answers from your class?
 Explain your thoughts to one of your classmates.
3. Think of some reasons why it is sometimes necessary to collect information from a small sample of people, instead of asking everyone in a school/city/country.
4. Now imagine you are going to select a sample of students from the whole school.
 Which sampling method would you use? Why?

Round Up

Organizations everywhere always need information from large groups of people, and this is exactly how they find it — by using small samples. But not all samples are equally good — you probably saw that for yourself. This Section tells you all you need to know about sampling.

Lesson 5.4.1

Section 5.4
Using Samples

You've seen lots of ways to analyze data sets, but this Section is about where that data comes from in the first place. For example, to find out how much time people in California spend watching TV, you could (in theory) ask them all. But it's much cheaper and easier to ask just some of them instead. The difficult thing is to know who to ask.

Data Is Sometimes Collected from a Whole Population

A **population** is the group of people or objects you want to gather information about. It could be everyone in your class, all the cars in California, all children in the U.S.A., or maybe all the animals in the world.

If a population is **small**, you could collect data about **every** person or object. For example, if you wanted to know how long each student in a class of 35 watched TV for one evening, you could ask **every student in your population of 35**.

Sometimes a Sample Is Used

If the population is **big**, it would be very **time consuming** and **expensive** to examine the whole population. Sometimes the size of a population isn't even known and it would be impossible to find every member.

For example, if you wanted to know how long each student in all of **California** watched TV for one evening, you couldn't ask every member of your population — you'd need to find every student in California.

What you can do is use a smaller part of the population called a **sample**.

A **sample** is part of a population.

The idea is that the sample will give you the **same sort of information** that the entire population would have given if you'd asked.

Example 1

A bank would like to find out what new services its 20 million customers would like it to offer.

The bank programs its ATM machines one Saturday morning with a questionnaire customers must complete before they can get any money.

Identify the population of this survey and the sample used.

Solution

The **population is all 20 million of the bank's customers** — these are the people the bank wants to know about.



The **sample consists of the customers who want to withdraw money from ATM machines on that Saturday morning** — these are the people the bank actually asks.

California Standard:

Statistics, Data Analysis, and Probability 2.1
Compare different samples of a population with the data from the entire population and **identify a situation in which it makes sense to use a sample.**

Statistics, Data Analysis, and Probability 2.4
Identify data that represent sampling errors and **explain why the sample (and the display) might be biased.**

What it means for you:

You'll learn what a sample is and why you might sometimes need to use one. You'll also see some things that you have to watch out for when using samples.

Key words:

- population
- sample
- bias
- claim

Check it out:

To survey the entire population, the bank would need to wait until every single customer had used an ATM machine — which is likely to be a long time.
Also, there would be so much data produced that it could be expensive to analyze.

Guided Practice

For each situation, identify the population and the sample.

1. A researcher wants information about the giardia present in streams in Orange County. He takes 8 one-liter containers of water from a stream in Santa Ana to conduct some tests on the giardia present.

2. To find out about the behavior of black bears in northern Washington, some were captured and fitted with radio collars. These bears were then released back into the wilderness and their movements tracked by monitoring the radio collars.

3. A person planning a party is looking for a band. To decide whether one particular band is suitable, she listens to 10 songs at random from the 50 songs the band play on occasions such as weddings.

Samples Should Represent the Entire Population

A sample should be **representative** of the **entire population**. This means it should contain the same kinds of people or objects as the population. That way, you should get the same information from the sample as you would have gotten from the population.

A sample that **doesn't contain** certain types of people or objects that **do** exist in the population is **biased**.

Check it out:
You have to make sure your sample is big enough too. Generally, bigger is better — a sample of 6 people out of a population of 10,000 won't give very useful results.

Example 2

An experiment into how children react to a certain vitamin supplement was conducted using 5000 boys from all over the USA.

Explain why this study is biased.

Solution

The **sample** contains **only boys**, but the **population** contains both **boys and girls**. The sample is not representative of the population.

Example 3

A study is carried out to discover whether residents of a town think a road should be widened. A questionnaire was mailed to every household in the town, and 97% of the replies were against the road widening.

Explain why this study may be biased.

Solution

It's impossible to tell if the sample is **representative of the whole town**.

People with **strong views** (such as those who live on the road to be widened) are more likely to reply to this kind of survey. People with less strong views may be less likely to respond, but they are still part of the **population**, and so their views are still important.

Check it out:
The population in this study is all the residents in the town. The sample used is the people who returned their form.

Check it out:
This kind of sample is called self-selecting. The people in the sample have chosen to be in it.

Claims made about the population based on **biased** samples are **invalid**.

Example 4

A country music radio station in California wants to find out what kind of music Californians prefer to listen to. It asks listeners to call in and vote for their favorite type of music. Later in the day, the radio DJ announces, "Californians' favorite music is country music."

Explain why the DJ's claim is not valid.

Solution

The population is all Californians. The sample consists of people who are listening to that particular country music station (who are likely to be country music fans).

People who dislike country music are probably not represented in the sample, making the **sample biased**. Results from this sample **cannot be applied to all Californians**, and so the claim is invalid.

Guided Practice

4. A survey is conducted to find out how Americans spend their leisure time. A sample of 1000 people are telephoned on a Saturday evening. It's found that the majority of people spend their leisure time watching TV at home. Describe why this sample is biased.

5. A study into the jobs of Californian people was carried out by knocking on all the doors on three Sacramento streets one Monday sometime between 8 a.m. and 5 p.m. and interviewing the householder. Describe why the results of the study would be invalid.

Independent Practice

1. Explain how a sample is different from a population.

In the studies described in Exercises 2–4, explain whether you would collect data from the entire population or a sample.

2. A study to find the average salary of workers in the United States.
3. A study to find the median height of students in a class.
4. A study to find the price of movie tickets in the United States.

Identify part(s) of the population underrepresented in these situations.

5. A survey collects responses on an internet website.
6. Data on traveling habits is collected at train and bus stations.

7. To gather opinions about a concert, the first 25 people leaving the hall were interviewed. Explain why claims based on this sample about what the whole audience thought would not be valid.

Now try these:
Lesson 5.4.1 additional questions — p455

Round Up

If the population is big, it makes sense to use a sample. But you have to take care how you select your sample — there are different ways of doing this, and they're not all equally good in all situations.

Lesson 5.4.2

Convenience, Random, and Systematic Sampling

If your population is too big to gather data about each member, you need to use a sample. The idea of sampling is to learn about a whole population by studying only a part of it. So it's important that a sample is chosen so that it accurately represents the population.

Convenience Sampling — Not Usually Representative

Convenience sampling is used because it's **easy** — **not** because it's best.

> **Convenience Sampling**
>
> In convenience sampling, each person or object is chosen because they are **easy to access**.

For example, you might want to find out what **people in California** think about an issue. One way to get some answers would be to go to your local mall and ask some **friendly-looking** people some questions.

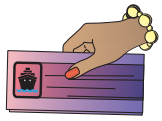
This is convenience sampling. It might be a useful way to get a few opinions, but the sample **won't be representative** of people in California.

That's the disadvantage of this method — samples aren't **representative** of the population, meaning you can't apply results to the whole **population**.

Example 1

A tour company wants to know how satisfied passengers on its cruise are. It interviews 100 passengers sitting in the ship's bar one lunchtime.

What kind of sampling is this?
Describe the advantages and disadvantages.



Solution

This is **convenience sampling**. The sample will be easy to find but **biased**, since the bar will be more attractive to certain people — such as those without children. This means the company couldn't claim that the results apply to the population (all the passengers).

Guided Practice

Exercises 1–2 describe convenience sampling.
Give one example of how each sample may be biased.

1. A running club wants to know the median age of runners across the USA. It finds the median age of its members.

2. A teacher wishes to find the average height of 6th grade students in a school. She asks the first 20 students to arrive for her class.

California Standard:

Statistics, Data Analysis, and Probability 2.2

Identify different ways of selecting a sample (e.g., convenience sampling, responses to a survey, **random sampling) and which method makes a sample more representative for a population.**

Statistics, Data Analysis, and Probability 2.4

Identify data that represent sampling errors and **explain why the sample (and the display) might be biased**.

What it means for you:

You'll find out about some methods of selecting a sample of a population, and think about the advantages and disadvantages of each.

Key words:

- population
- sample
- convenience sampling
- random sampling
- systematic sampling

Check it out:

In Example 1, recording the ages of the first 100 people to board might produce a biased sample too. Certain age groups might be more likely to arrive at the ship in good time and board earlier.

Check it out:

Many businesses use convenience sampling because it is easy to sample their own customers.
And researchers collecting information from people at the mall often choose the easiest subjects — approaching those that walk nearest them and appear to be friendly.

Random Sampling — Should Be Representative

Random Sampling

In random sampling, each person or object in the population has an **equal chance** of being selected.

Random sampling is difficult because, for large populations, it's very hard to make sure that every member of the population could be selected.

For example, one way to get a random sample of people in California would be to get a list of **everybody** in California. Then you'd need to randomly pick people on that list. Then you'd need to go and talk to them — no matter where they lived. You couldn't decide **not** to visit someone because they lived a long way away, or because they didn't want to talk to you. That would mean the sample **wasn't random**.

Don't forget:
The size of a sample is important too.

But... the big advantage of random sampling is that as long as your sample is big enough, it should be **representative** of the population.

Example 2

A hotel wants to see if, in general, its customers are satisfied with the service they receive. They devise two possible ways to obtain a sample.

Method 1: Interview the guests staying in the rooms nearest the lobby.
Method 2: Select guests at random and interview these people.

Say whether each method is convenience sampling or random sampling. Which method do you think would be better to use, and why?

Solution

Method 1 is **convenience sampling**. The sample may be **biased**.
The area near the lobby area may be very noisy, for example.
This will mean the hotel will get more complaints than is representative of the whole hotel (or fewer, if the area near the lobby is very quiet).

Method 2 is **random sampling**. Each customer has an equal chance of being selected — it doesn't matter where in the hotel they have stayed.
This method would be the better one to use, as the sample should be representative of all the hotel's guests at that time.

Guided Practice

3. Which one of the following is true of random sampling?
A. Random sampling is a quick way to obtain data.
B. It should produce samples that are representative of the population.
C. It is easy and quick to do.

4. Describe how random sampling could be used to select a representative sample of 50 of a school's 6th grade students.

Systematic Sampling Means Using a Set Pattern

Systematic Sampling

In systematic sampling, people or objects are chosen by a set pattern.

An example of systematic sampling would be collecting data from **every fifth student** entering the auditorium. It's usually a good way to pick a sample, but it has a big disadvantage...

Suppose every **100th item** on a factory assembly line is **faulty** because of a computer glitch. And suppose that the factory's quality-control team uses **systematic sampling**, and tests every **300th item** produced.

There are two possibilities:
(i) a faulty item is never checked, or (ii) the only items checked are faulty.
Neither of these would give a true picture of the actual situation.

But in many situations, systematic sampling is usually easy and effective.

Independent Practice

1. What is the convenience sampling method?

2. What does it mean if a sample is collected using random sampling?

3. The manager of a clothes store wants to know how much the store's customers spend on clothes each month. He interviews the first 100 shoppers in line for a sale. What kind of sampling is this?

4. Which of the following is NOT one of the reasons why you might want to use convenience sampling?

A. The data is easily available.
B. It can be less expensive than some other sampling methods.
C. Convenience sampling can be a way to quickly obtain data.
D. It always produces samples that accurately represent the populations.

5. What is systematic sampling?

6. The math teacher has told students to take a sample from a population of 150 items. What type of sampling does each student use?

Sarah writes the numbers 1–150 on slips of paper and puts them in a paper bag. She then draws out the numbers 32, 26, 116, 53, and 141, so she uses the 32nd, 26th, 116th, 53rd, and 141st items for her sample.

Micah uses every 10th item in the population for her sample.

Miguel chooses the first 10 items in the population.

Now try these:
Lesson 5.4.2 additional questions — p455

Round Up

So now you have seen three methods of selecting a sample from a population — convenience, random, and systematic. There are advantages and disadvantages to each, of course.

Lesson 5.4.3

Samples and Accuracy

California Standards:

Statistics, Data Analysis, and Probability 2.1

Compare different samples of a population with the data from the entire population and identify a situation in which it makes sense to use a sample.

Statistics, Data Analysis, and Probability 2.4

Identify data that represent sampling errors and explain why the sample (and the display) might be biased.

What it means for you:

You'll see that no matter how carefully you select your sample, it's unlikely to represent the population totally accurately.

Key words:

- population
- sample
- mean
- sampling error

Don't forget:

You find the mean of a data set by adding up all the values and dividing the total by the number of values. See Lesson 5.1.2 for more information.

When you select a sample, you need it to be as representative as possible of the whole population. A sample that's representative should have a similar mean, mode, median, and range to the population. Most likely, they won't be exact matches, of course — but they should be pretty close.

Sampling Errors Should Hopefully Be Small

Collecting data from a **sample** is usually easier and cheaper than collecting data from an entire **population**. But you need your sample to represent the population as accurately as possible.

You can check how accurately your sample represents the population by checking its **statistics** — for example, the mean, mode, median, and range.

> A **sampling error** for the mean is the difference between the **mean of the sample** and the **mean of the population**.

You can also find sampling errors for measures such as the mode and median. The **smaller** the **sampling error**, the more accurately the sample represents the population.

Example 1

A state with 50 gymnastics teams reports the number of medals they each won in state competitions this year.



Medals won: 1, 12, 3, 16, 20, 9, 8, 5, 2, 7, 36, 7, 7, 9, 19, 7, 8, 27, 2, 6, 4, 9, 7, 8, 7, 6, 8, 6, 9, 10, 8, 19, 12, 8, 9, 9, 2, 5, 8, 10, 7, 2, 6, 6, 7, 6, 29, 8, 5, 5

A sample of 5 was randomly selected from the population. Find the sampling error for the mean.
Sample: {10, 16, 8, 8, 7}

Solution

First, you need the **population mean**.

This is $\frac{1+12+\ldots+5+5}{50} = \frac{456}{50} = \mathbf{9.12}$ **medals**.

Next, find the **sample mean**.

This is $\frac{10+16+8+8+7}{5} = \frac{49}{5} = \mathbf{9.8}$ **medals**.

So the **sampling error** is $9.8 - 9.12 = \mathbf{0.68}$ **medals**.

Example 2

Find the sampling error of the median number of medals won (using the data in Example 1) if the following sample is used: {8, 27, 6, 8, 9}.

Solution

To find the **population median**, you have to put all 50 values in order. Then the median will be midway between the 25th and 26th values. In fact, the median for the population is **7.5 medals**.

Now you need the **sample median**.
So put the values in order: 6, 8, 8, 9, 27.
The median is the middle value — this is **8**.

So the sampling error for the median is 8 – 7.5 = **0.5 medals**.

Don't forget:
Subtract the population median from the sample median to get the sampling error.

Guided Practice

Below is a list of the distances (in miles) traveled to school by students in a class of 28 students.
1, 8, 3, 5, 2, 5, 3, 3, 2, 7, 4, 2, 4, 3, 5, 6, 1, 1, 5, 3, 2, 3, 2, 1, 4, 1, 2, 1

The following sample is used: {4, 1, 8, 2, 3, 6, 3, 4}

1. What is the population mean?
2. What is the sample mean?
3. What is the sampling error for the mean?

4. What is the population median?
5. What is the sample median?
6. What is the sampling error for the median?

7. Repeat Exercises 2–6 for the sample {5, 4, 2, 6, 1, 3, 2, 4, 1}. Comment on your results.

Different Samples Give Different Sampling Errors

If you take different **samples** from the same population, then you'll probably get different sampling errors.

Example 3

Find the sampling errors for the mean using the data in Example 1, if the following two samples are used.
Sample 1: {5, 5, 2, 20, 8}, Sample 2: {9, 8, 6, 8, 27}

Solution

For Sample 1, the sample mean is $\frac{5+5+2+20+8}{5} = \frac{40}{5} = 8$.

This gives a sampling error of 8 – 9.12 = **–1.12 medals**.

For Sample 2, the sample mean is $\frac{9+8+6+8+27}{5} = \frac{58}{5} = 11.6$.

This gives a sampling error of 11.6 – 9.12 = **2.48 medals**.

Don't forget:
It's completely normal to get different sampling errors from different samples — it's not a mistake.

Check it out:

There are many different factors that can affect the sampling error — the size of the sample is just one of them. You can still have a large sample that's biased and unrepresentative. However, using a fairly large, randomly selected sample should reduce the risk of this.

Check it out:

In Example 4, the larger sample gave a smaller sampling error. But notice that the smaller samples used in Examples 1 and 3 gave smaller sampling errors. That's statistics for you.

If you picked lots of small samples and lots of big samples, then you'd generally expect the bigger samples to have smaller sampling errors. But it's still possible that you could pick a few small samples with really small sampling errors, or a few big samples with quite big sampling errors.

Now try these:

Lesson 5.4.3 additional questions — p455

The aim when you pick a sample is to make your sampling errors **as small as possible**.

Generally, the **bigger your sample**, the **smaller your sampling error** is likely to be. (However, the bigger your sample, the more expensive and difficult it is to collect the data.)

Example 4

For the data in Example 1, find the sampling errors for the mean if the following randomly generated samples are used.
Sample 1: {2, 6, 6, 8, 9},
Sample 2: {2, 2, 2, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 12, 19, 19, 29, 36}.

Solution

For Sample 1, the sample mean is $\frac{2+6+6+8+9}{5} = \frac{31}{5} = \mathbf{6.2}$.

This gives a sampling error of 6.2 – 9.12 = **–2.92 medals**.

For Sample 2, the sample mean is $\frac{2+2+\ldots+29+36}{20} = \frac{211}{20} = \mathbf{10.55}$.

This gives a sampling error of 10.55 – 9.12 = **1.43 medals**.

Notice that the **larger sample** gave a **smaller sampling error**.

Guided Practice

8. Find the median of the samples given in Example 4.
9. What is the sampling error for each of these samples?
10. Is this what you would have expected? Explain your answer.

Independent Practice

A scientist measures the lengths of 300 fish, before dropping them into a pond. Later, three different samples of the fish are measured (in cm).
Sample A: {4, 5, 5, 5.5, 6, 6.5}
Sample B: {3, 5, 5.5, 5.5, 6, 6, 7.5, 8, 11}
Sample C: {4.5, 5, 5, 5.5, 7, 7.5, 7.5, 8, 10, 11}

1. Find the mean of each of the three samples.
2. The mean length of the population of fish is 5.99 cm.
Find the sampling error for each sample based on the mean.
3. Which sample most accurately represents the population?
Is this what you would have expected? Explain your answer.

4. Another sample of the fish is chosen using the first ten fish the keeper manages to catch. What type of sampling is this? Would you expect this sample to accurately represent the population? Explain.

Round Up

Sampling isn't likely to give you absolutely perfect results, but if it's done carefully, then it's good enough for most purposes. In real life, picking a sample is a compromise — a bigger sample should make for a smaller sampling error, but bigger samples are more expensive and difficult to use.

Lesson 5.4.4

Questionnaire Surveys

You can get lots of data from people by asking them questions, but only if they choose to answer them. This isn't perfect, but it's the only way of getting some types of information.

People Have to Choose to Answer a Survey



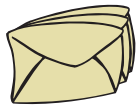
A lot of information is collected by contacting people by **phone**, **mail**, or **in person** and asking them to provide some data.

This is sometimes called **self-selected sampling**, because the people must **agree** to give the information. Self-selected samples aren't always ideal — a **random** sample might be better in some ways. But you can't force people to answer questions if they don't want to.

Even though it's not a perfect way to gather information, this kind of self-selected sample can still provide a lot of very useful information.

Example 1

A company wants to find out which cleaning products are used in California households. It sends out questionnaires in the mail to all the customers listed in their database.



Identify the population and the sample used.
What are the benefits and problems of collecting data in this way?

Solution

Population: All the households in California.
Sample: Those people listed in the company's database who choose to fill in and return the questionnaire.

Benefits: The company **learns useful information** such as which products are most popular, and the location of customers who purchase certain products. It also learns which households are more likely to complete other surveys in the future.

Possible problems: Because the sample is **self-selecting**, and because the only people in the sample are previous customers of the company, the sample is **unlikely to be representative** of the population. Also, the costs of printing and mailing are likely to be high, and a lot of this money will be wasted, given that most people will probably not respond.

Instead of sending out questionnaires, a lot of organizations these days use the **internet** to gather information. As with any method, there are benefits and disadvantages.

California Standard:

Statistics, Data Analysis, and Probability 2.2
Identify different ways of selecting a sample (e.g., convenience sampling, **responses to a survey,** random sampling**) and which method makes a sample more representative for a population.**

Statistics, Data Analysis, and Probability 2.3
Analyze data displays and **explain why the way in which the question was asked might have influenced the results obtained** and why the way in which the results were displayed might have influenced the conclusions reached.

What it means for you:

You'll find out some good and bad points of surveys, and some ways to make your data as useful as possible.

Key words:

- population
- sample
- survey
- self-selected sampling
- convenience sampling
- random sampling

Check it out:

When a survey is sent out in the mail, very few people will probably respond.

Check it out:

Surveys can be used to collect data on pretty much anything — such as age, gender, household income, number of cars owned, number of family members...

Example 2

The same company as in Example 1 decides to use the internet to gather the information it needs. It sends e-mails to all the customers whose e-mail addresses it has. The e-mail tells customers that if they fill in an online questionnaire, they have a chance to win $1000.

Identify the population and the sample used. What are the benefits and problems associated with collecting data in this way?

Solution

Population: All the households in California.
Sample: The customers who:
(i) have made a purchase from the company, (ii) gave their e-mail address, and (iii) who responded to the e-mail.

Benefits: The survey provides the company with information at **very little cost** — there are no mailing costs, and the data can be automatically analyzed by computer.

Possible problems: The sample **won't be representative** of the population. The sample is **self-selecting** (so is likely to include a lot of people with strong opinions), and won't include people **who don't have access to the internet**.

Guided Practice

A campaign group wants to find out if taxpayers would be willing to pay higher taxes if a new community swimming pool were promised for their area.
1. Describe how information could be gathered from local taxpayers.
2. What would the population and sample be in your example?
3. Describe some advantages and disadvantages of your method(s).

A car dealer wants to collect data about when local people are planning to buy a new car. She organizes a prize drawing, which people enter by filling out a form with their name, address, and when they expect to buy a new car.
4. What are the population and sample in this example?
5. Describe some advantages and disadvantages of this method.

Questions Can Be Biased Too

Surveys can give inaccurate information if a **sample** is **biased**.
But information collected by asking people **questions** can also be **biased** if the questions **encourage** the person to give a **particular answer**.

A **biased question** is one that encourages people to answer in a particular way.

Check it out:
Biased questions are also called "leading questions."

Check it out:
When a claim is based on the responses to biased questions, it is invalid. The claim can't be trusted, since there is no real, unbiased evidence to back it up.
See Lesson 5.5.1 for more information about claims.

People often ask biased questions if they are collecting data to try to back up one of their **opinions**. For example, suppose someone is in favor of banning smoking from public parks...

Example 3

Explain why the following question is biased.

> Shouldn't smoking stinky cigars and unhealthy cigarettes be banned from all public parks?

Solution
The question is worded so that the strong negative opinion about smoking in public makes it difficult for a person to disagree.

These are called **leading questions**. They **lead** the person answering toward a particular reply.

Example 4

Explain why the following questions are biased.

(i) Experts say that you should chew your food slowly and take at least 30 minutes for a meal.
How much time do you think students should get for lunch break?

(ii) A lot of models and movie stars have a yoga instructor. Don't you think it would be better to offer yoga instead of basketball in middle school?

Solution
(i) A response of at least 30 minutes or more is encouraged by stating a related "fact."
(Beware... it may not be a genuine fact. It doesn't say who the experts are, or if any other people disagree. Only one opinion is given.)

(ii) Many people think models and movie stars are attractive and successful. This question suggests that yoga is one of the causes of their success. This may make people more likely to agree that yoga should be offered instead of basketball.

Guided Practice

Explain why the questions below are biased.
6. "Don't you think we should brighten up the halls by painting over that boring tan color?"
7. "Painting the school will cost a lot of money that could be spent on more important things. Don't you agree that painting the school can wait for another couple of years?"

Write Survey Questions So They're Not Biased

You need to write survey questions so that they **don't** suggest that a certain reply is correct. You shouldn't reveal your opinions (or anyone else's) when asking questions.

Here are the questions from earlier, written in a **less biased** way.

Are you for or against banning smoking in public parks?

How much time do you think students should get for lunch break?

Do you think it would be better to offer yoga or basketball in middle school?

Guided Practice

A school committee is surveying students about whether a salad bar should be added to the cafeteria.

8. Write a biased question in favor of adding a new salad bar.

9. Write a biased question against adding a new salad bar.

10. Write an unbiased question to gather opinions on a new salad bar.

Independent Practice

For Exercises 1–2, explain why each sample may not be representative of a population consisting of all the adults in a particular town.

1. A researcher asks shoppers entering or leaving a grocery store if they can answer some questions about their purchases.

2. A receipt from a department store gives a website where customers can go online to complete a questionnaire about public transportation.

For Exercises 3–6, decide whether the question is biased. If it is biased, explain why and suggest a better way of asking the question.

3. "Should the driving age in California be raised to 21?"

4. "Should we reduce the number of car accidents by fining drivers who talk on their cell phones while driving?"

5. "A community center would give our children a safe place to play. Can we count on your vote this Tuesday for keeping our children safe?"

6. "Do you think mining companies should be allowed to ruin our wilderness areas and kill innocent animals?"

7. 300 middle school students were asked, "Isn't it unfair that people under 18 can't vote for president?"
As a result, it was claimed that 95% of middle school students thought the voting age should be lowered. Is this claim valid? Explain.

Now try these:
Lesson 5.4.4 additional questions — p456

Round Up

Surveys are a good way of collecting data from people. You've got to be really careful how you word the questions, though. You also have to make your sample as representative as possible — this isn't always easy, since the way you carry out your survey means certain people are more likely to provide information than others. But even though it's difficult, surveys like this can be incredibly useful.

Lesson 5.5.1

Section 5.5
Evaluating Claims

California Standard:

Statistics, Data Analysis, and Probability 2.5

Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.

What it means for you:

You'll look at data intended to support claims and decide if the claim is valid.

Key words:

- claim
- opinion
- valid
- invalid

You're likely to be bombarded by claims every day. Claims are presented as facts — but they might not be. For instance, advertisements often claim that one product is better than another.

You can only really tell whether a claim is true or not if you examine the evidence.

A Claim Is a Statement Presented as a Fact

Claims are often reported in newspapers and on television and radio. These are examples of claims:

> The average American eats pizza at least once a week.

> More people choose this toothpaste than any other.

> In 100 years, there won't be any glaciers left on Earth.

Claims are **different** from opinions. A claim is presented as a **true fact**. An opinion is just someone saying what they believe, such as "I think this brand of cola tastes best."

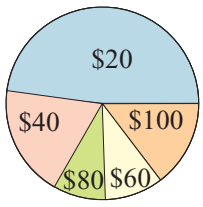
Claims Are Often Supported by Data

Data is often used to provide **evidence** for a claim.
You need to look carefully at the data to see if it supports the claim.

Example 1

A grocery-store owner claims that \$20 is the amount most often withdrawn from the ATM outside his store during one month.

Does this circle graph from the bank's monthly report, which shows data about that ATM, support his claim?



Solution

This circle graph does show that the **modal amount** withdrawn was **\$20**.

So the circle graph **does support** the store owner's claim.

This **doesn't** show that \$20 is **always** the amount withdrawn most often. Next month's data might be different.

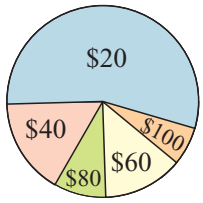
So if the store owner had claimed that \$20 was **always** the amount most frequently withdrawn, then you would need to look at more evidence.

Don't forget:

The modal amount (or the mode) is the amount that occurs most frequently. See Lesson 5.1.1 for more information.

Example 2

A different store owner claims that $20 is the **mean** amount withdrawn from the ATM outside her store during the same month.



Does this circle graph from the bank's monthly report, which shows data about that ATM, support her claim?

Solution

You need to read the claim very carefully.

This time, the claim is that $20 is the **mean** amount. This **can't be true**, since $20 is the smallest amount withdrawn, and all the other amounts will make the mean amount greater than $20.

So the circle graph **does not support** this store owner's claim.

Don't forget:

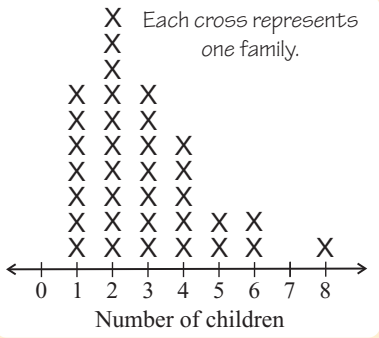
Remember... In Example 2, values higher than $20 will push the mean up. And there are no values less than $20 to bring the mean back down. See Lesson 5.2.2 for more information.

Guided Practice

1. Is this statement a claim or an opinion? "I think the red team is a better team than the blue team." Explain your answer.

2. It's claimed that most families using a swimming pool on a Saturday morning have exactly 2 children.
Does the line plot on the right support this claim?

3. Give one example of a claim that would be supported by the line plot.



Sometimes the Evidence Is Not Clear

Sometimes, some statistics may support a particular claim, but other statistics will not.

People trying to convince or persuade someone will very often use just the statistics that **support** their claim. They may **ignore** statistics that don't support their claim, or which make their claim look untrue.

For example, companies make claims to try to **persuade** you to buy their products. They'll want to use statistics to make their product look like it's **better** than the competitors'.

One way of doing this is by using a particular measure of central tendency, depending on what claim you want to make. Maybe they want to make the typical value sound **high**. Or maybe they want to make it sound **low**.

Don't forget:

Measures of central tendency are the mean, median, and mode. See Lessons 5.1.1 and 5.1.2 for more information.

Example 3

A newly opened health club wants to attract some new young members. Its current members' ages are: 25, 25, 37, 45, 53, 54, 55, 57, 63, 67.

It wants to advertise, making the claim that the typical age of its current members is low. To achieve this, its advertisement makes the claim that "There are more 25-year-olds at the club than any other age."

Does the data support this claim? Comment on the claim generally.

Solution

The modal age is 25, so **the data does support the claim**.

However... the **mean** age is $\frac{25+25+\ldots+63+67}{10} = \frac{481}{10} = \mathbf{48.1}$.

And the **median** age is **53.5**.

So if the health club had used any other measure of central tendency, the typical age would have **sounded a lot higher**.

Here, the health club wanted to make the typical age of its members **sound** low in the advertisement, so used the **only** measure of central tendency that would do this.

The club's claim is not untrue, but it probably is misleading.

Statistics aren't always used in **deliberately misleading** ways. Sometimes, whether a claim is supported by data can be unclear for different reasons.

Example 4

At the swim meet, the team swam in three relay races.
The times for each team member in each race are shown below.

	Race 1	Race 2	Race 3
Lucas	22.5 s	25.7 s	19.5 s
Latoya	23.0 s	19.7 s	22.1 s
Jorge	27.5 s	26.2 s	27.4 s
Anna	23.1 s	20.2 s	21.4 s

Lucas claims that he's the fastest swimmer in the relay swim team. His sister, Sam, claims that Lucas is not the fastest.

Whose claim do the results support?

Solution

It's hard to tell just by looking at these results if Lucas is the fastest. He swam fastest in Race 1 and Race 3, but not in Race 2.

One way to **analyze** the results is to find each person's mean time.

Lucas: (22.5 + 25.7 + 19.5) ÷ 3 = **22.6 s**
Latoya: (23.0 + 19.7 + 22.1) ÷ 3 = **21.6 s**
Jorge: (27.5 + 26.2 + 27.4) ÷ 3 = **27.0 s**
Anna: (23.1 + 20.2 + 21.4) ÷ 3 = **21.6 s**

Both Latoya and Anna had **faster average times** than Lucas.
So Sam could say that **the results support her claim**.

On the other hand, **no one swam faster than Lucas's time of 19.5 s**.
So Lucas could say that **the results support his claim**.

Don't forget:

See Lesson 5.1.4 for more information about comparing data sets.

In cases like this, it's not clear whose claim the data supports.
The best thing is probably to say whether you're interested in the **fastest average time**, or the **fastest single time**.

Or you could look more closely at the data. Did something happen to Lucas in the second race to slow him down? Or perhaps he was doing a different stroke — maybe Lucas is fastest at one stroke, but not at another.

Guided Practice

A babysitting company has six sitters of ages 13, 13, 14, 15, 15, and 45. Use this information for Exercises 4–7.

4. Find the mean, median, mode, and range of the babysitters' ages.

5. The company claims that the typical age of its babysitters is 19. Which measure of central tendency supports this claim?

6. The claim in Exercise 5 is valid, but why is it misleading?

7. Make two claims that better represent the typical babysitter's age.

Independent Practice

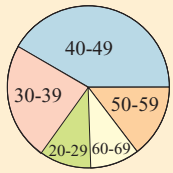
Use the terms **data**, **valid**, **invalid**, or **claim** to complete the sentences in Exercises 1–3.

1. A _____ is a statement that is presented as a fact.

2. A claim is considered _____ if the _____ is incorrectly interpreted.

3. A claim is usually _____ if there is no _____ to support it.

4. Someone reading the circle graph on the right made this claim: "Most people on the school board are aged 40–49." Explain why this claim is invalid. Suggest why the person may have thought that this claim was true.



Here are the ages of the counselors at a summer camp:
{19, 20, 22, 23, 20, 55, 20}

5. Find the mode, median, and mean for this data set.

Use your statistics from Exercise 5 to say whether the claims in Exercises 6–10 are valid or invalid. For each valid claim, say which statistic supports the claim.
Are any of these claims valid but misleading? Explain your answer.

6. Most counselors are aged 24.

7. The typical age of the counselors is over 25.

8. Half the counselors are older than 24.

9. Half the counselors are 20 or younger.

10. More than 14% of the counselors are over 30.

Now try these:
Lesson 5.5.1 additional questions — p456

Round Up

People often make claims. Often their claims are valid and backed up by data. But sometimes the data doesn't support the claims or has been misinterpreted. Either way, the claim won't be valid.

Lesson 5.5.2

Evaluating Displays

If people want to convince you of something, then they will probably present their data in such a way that it emphasizes what they want you to believe. One way to do this is with graphs and charts.

California Standard:

Statistics, Data Analysis, and Probability 2.3

Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and **why the way in which the results were displayed might have influenced the conclusions reached.**

What it means for you:

You'll see how data can be displayed in ways that may mislead people.

Key words:

- data set
- display
- line plot
- mean
- median
- mode
- range

Check it out:

A "break in the scale" means the numbering on the axis doesn't start at zero. In the first graph in Example 1, the numbering starts at 70.

Don't forget:

Data can also be displayed as circle graphs, line graphs, or bar graphs.

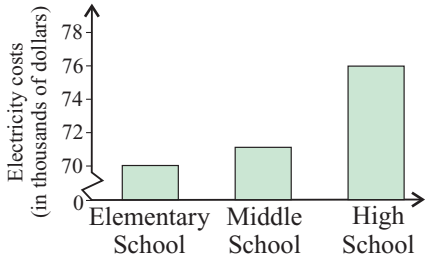
A Break in the Axis Can Create a Visual Effect

Putting a **break** in the vertical axis of a graph can change the impression given by a graph.

Example 1

At a school-board meeting, a member claims that electricity costs in local high schools are much higher than for elementary and middle schools. He uses the graph below to support his claim.

Why might someone complain that the graph is misleading?

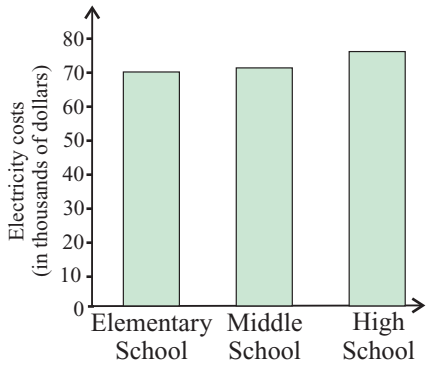


Solution

The break in the vertical scale makes the bar on the right appear much longer than the other bars.

If the vertical scale starts at zero, the difference appears to be much less dramatic.

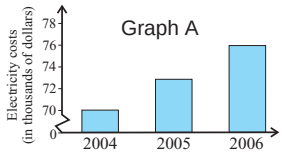
The break in the scale on the first graph emphasized the difference in height of the bars.

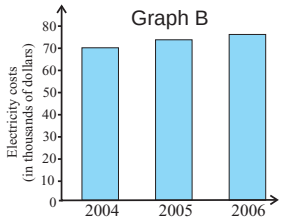


The same idea can make things look like they're rising or falling very quickly.

Example 2

Which of these graphs makes it look like electricity costs are rising more quickly?





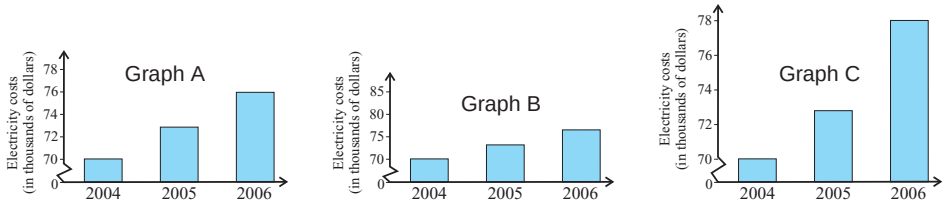
Solution

Graph A, because the difference in height between the bars is emphasized by the break in the scale. This makes it look like costs are rising quickly, and is possibly misleading.

Using a **different scale** can achieve the same effect.

Example 3

Which graph makes electricity costs look to be rising most quickly?



Solution

This time, the effect is achieved by using different scales on the vertical axis. (All three graphs have vertical scales starting at 70.)

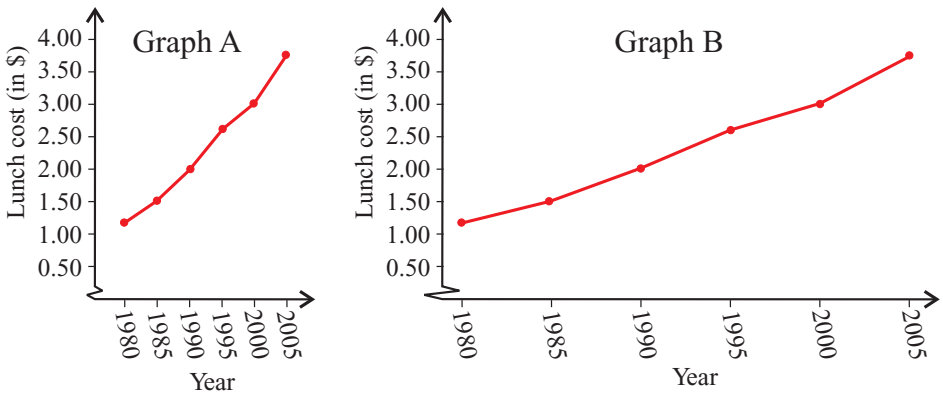
Graph B looks like it's rising least quickly. This is because its vertical scale is marked in intervals of 5, rather than in intervals of 2 as in Graphs A and C.

Graph C looks like it's rising most quickly, because it is marked in intervals of 2, and it has been stretched vertically.

The next example shows how the spacing on the **horizontal axis** can be changed to emphasize a particular viewpoint.

Example 4

Which of these graphs might someone use to support the claim that school lunch prices have risen steeply in the past 25 years? Explain.



Solution

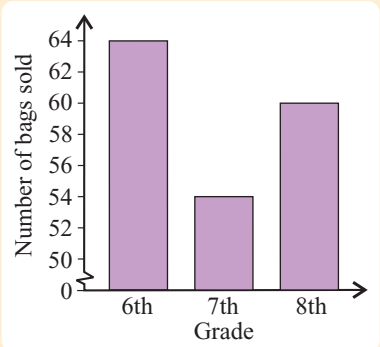
Graph A — it appears much steeper than Graph B.

Graph A gives the impression of a greater rate of increase because the line is **steeper**. In Graph B the line appears flatter, so the increase in prices does not appear to be so drastic.

Both graphs are drawn from the same data, but the years on the horizontal axis of Graph B are spread **further apart**.

Guided Practice

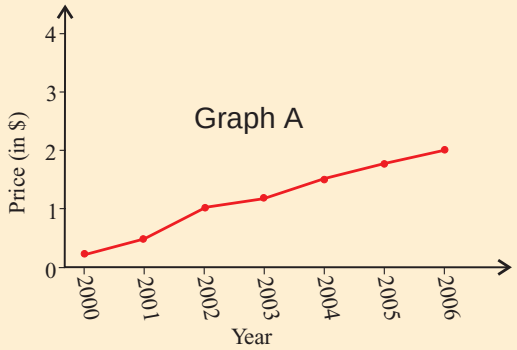
This graph shows the number of bags of popcorn sold by students in grades 6, 7, and 8 to raise money for a charity.

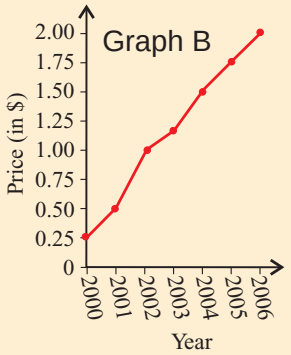


The graph was produced by grade 6.

1. Students in grade 7 thought this graph was misleading. Suggest why.
2. Draw a graph to show the same information more clearly.

3. Which graph below makes the price of a school carnival ticket appear to increase more dramatically? Explain why.

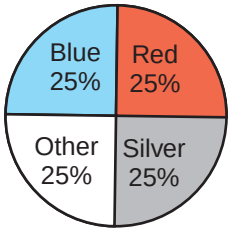




4. Suggest how someone could avoid getting the wrong impression from a graph.

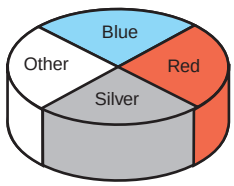
Things Can Be Emphasized in Other Ways Too

You've seen **circle graphs** before. Like any other kind of graph, the way they are presented can emphasize a particular viewpoint.

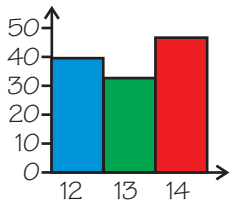


For example, take this circle graph showing how common different colors of cars are. All the different categories (blue, red, silver, and other) are **equally popular**.

But if you draw this in **three dimensions**, things look a little different. Suddenly, silver looks (at first glance, anyway) to be the most popular color, because more of the diagram is silver than any other color. The three-dimensional effect **emphasizes** the amount of silver cars.

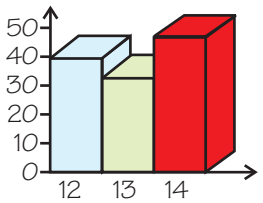


You can do the same thing with **bar graphs**. You can **emphasize** some bars but not others by using a three-dimensional graph.



Look at this graph showing how old the students who play basketball at one particular club are. There's a **good mix** of ages — you can see this from a quick glance at the graph.

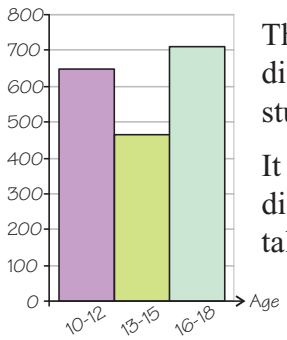
Now look at the same graph drawn in **three dimensions**.



Suddenly, red is the **dominant color** — you can see the **whole volume** of the red bar, but not of the others.

Check it out:
The colors of the other bars have also been made less bright. This also helps to emphasize the amount of red on the graph.

The **grouping** of data values can also give a misleading impression. For example, look at the graph below.

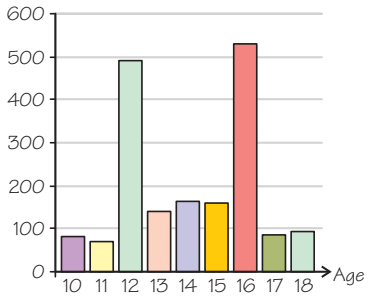


This graph shows the number of students of different ages who voted in the last election for the student council.

It looks like there is a good mix of students of different ages voting — none of the bars are much taller than the others, and none are much shorter.

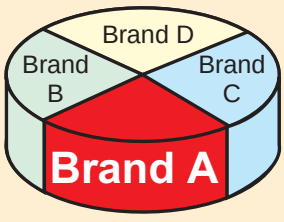
But look what happens if you now draw one bar for each year of age.

It's now clear that the vote was dominated by 12- and 16-year-olds.



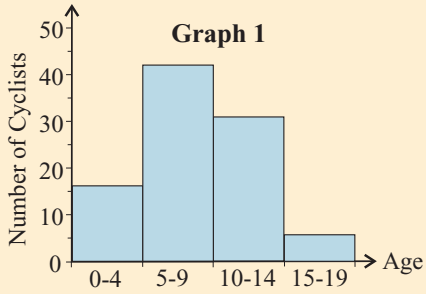
Guided Practice

5. Why might the manufacturer of Brand A want to use a graph like this one to compare how popular different makes of cars are?
6. Why might someone claim that this graph was misleading?
7. Redraw the graph to show the same information in a clearer way.



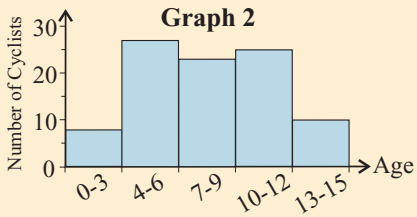
Independent Practice

For an annual children's bicycle race, the participants enter by age categories. Graph 1 on the right shows this year's entries.



1. Which age interval has the greatest number of participants?
2. Why can't you tell from the graph if any 19-year-olds entered?

Graph 2 below shows the same data, but with the ages grouped in a different way.



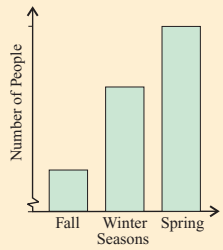
3. How do the new intervals used in Graph 2 change the overall impression of the ages of the competitors, compared to Graph 1?

Graph 1 appeared in the newspaper with the headline "5 through 9 year olds dominate the event."
4. What headline might have appeared if Graph 2 was printed in the newspaper instead?

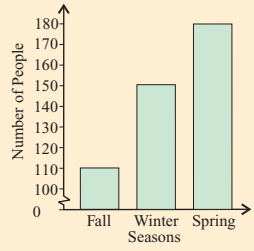
5. Give an example of a claim for which you would want to exaggerate the difference between the lengths of the bars of a bar graph.

6. Give an example of a claim which you would want to support by a line graph in which there appears to be less dramatic change over time.

The graph on the right appeared in the school newspaper in an article about the popularity of spring sports. It shows the attendance at school sporting events during different seasons.



7. Approximately how much more popular do sporting events in the spring seem to be, compared with sporting events in the fall?
8. How does your answer to this question differ if you use the graph below, which shows actual data values too? Explain why this is.



9. Redraw the bar graph using a vertical scale from 0 to 200 in increments of 50 without any break in the scale.
10. Describe the impression given by the graph you have drawn.

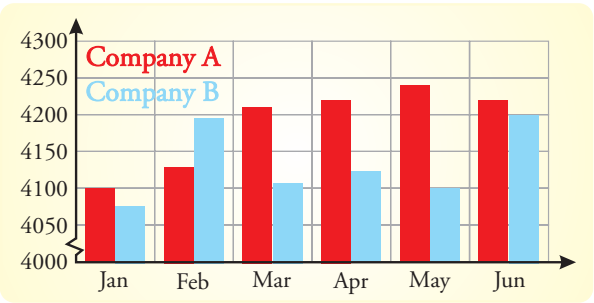
Now try these:
Lesson 5.5.2 additional questions — p457

Round Up

When you're looking at a data display, take care — it might be drawn in a misleading way. Always check the scales and think about how the data would appear if the display was drawn differently.

Chapter 5 Investigation
Selling Cookies

You work for a supermarket. A team of sales people from Company A are visiting. They say their cookies cost a little more than Company B's brand, but they sell far better. Below are parts of a report from Company A, and some sales data.



Sales Data Packs of Cookies sold in 6 months from Jan-Jun

Brand	Jan	Feb	Mar	Apr	May	Jun
Company A	4100	4136	4208	4220	4240	4222
Company B	4075	4198	4106	4123	4100	4200

- Typical monthly sales for Company A are 100 greater than for Company B.
- In every month since March, Company A cookies have sold more than Company B.
- According to our survey, 88% of people prefer Company A's cookies, and feel they are just as healthy as any other brand.

Part 1:
Look at the **graph** taken from the **Company A report** and the **sales data**. The sales people explain that the graph shows Company A's cookies selling in **far greater quantities** than Company B's.

What advice would you give to your manager about the graph?

Part 2:
Now look at the above excerpt from the **Company A report**, the **sales data**, and the **questions** asked in the Company A survey (on the right).

Are the claims in the report in any way misleading?

Appendix: Company A's Survey Questions
1) Would you agree that Company A's cookies are the tastiest cookies you can buy?
2) Our experts say that Company A's cookies are just as healthy as any other cookies. Do you agree that Company A's cookies are no less healthy than any other brand?

Extension
Your supermarket wants to do its own research, rather than trust Company A.

1) Describe two ways your company could use sampling to collect this information. Which method would you recommend? Why?
2) Write three unbiased survey questions that could be used.

Open-ended Extension
Conduct a survey to find out people's favorite: **food** OR **sport** OR **type of music**.

1) Write an **unbiased** survey question for your topic. Ask **20 people** the same question.
2) Now reword your question to favor a particular response. Ask a **different** set of **20 people** the new question. Did your question affect the responses?
3) Describe the sample of people chosen to answer each question, and the population they represent. Do you think your samples were representative of the population? Explain.

Round Up
Statistics can be used accurately, or they can be used in ways that may mislead people. Even if you'd never mislead anyone, it's good if you can see when other people are trying to mislead you.

Chapter 6

Probability

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Lesson 6.1.1

Section 6.1
Listing Possible Outcomes

California Standard:

Statistics, Data Analysis, and Probability 3.1

Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.

What it means for you:

You'll think about actions that can have more than one possible result. You'll list all the possible outcomes for particular situations.

Key words:

- outcome
- event
- action
- experiment
- combine

Check it out:

Think of an outcome as any possibility, and an event as a set of outcomes matching a particular condition.

You might have played probability games or board games at school or at home. If you have, then even if you didn't realize it at the time, you've worked with events and outcomes.

An Outcome Is the Result of an Action or Experiment

Many situations have more than one **possible outcome**.

Example 1

Sarah and her brother Nathaniel are trying to decide who gets to drive the car tonight. They decide to toss a coin.
How many different outcomes are possible when they toss the coin?



Solution

Possible Outcomes: **Heads** is one possible outcome.
Tails is one possible outcome.

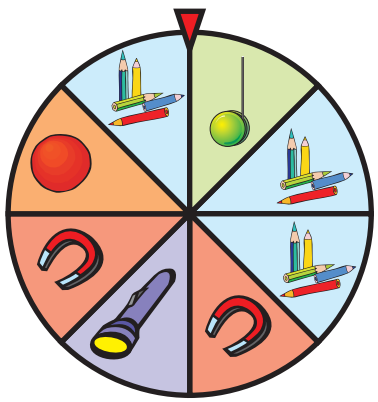


These are the **only** possible results.
There are a total of **2 possible outcomes** for the action of tossing a coin.

An **event** is a description that matches **one or more** possible **outcomes**.

Example 2

At a carnival, kids spin the wheel shown to win a prize. How many possible outcomes match the following events?
Event A: winning a flashlight
Event B: winning a magnet
Event C: winning coloring pencils



Solution

There are 8 sections on the wheel, so there are **8 possible outcomes**.

There is 1 section of the wheel that will let you win a flashlight, so **1 outcome** matches event A.

You can win a magnet by spinning either of 2 sections.
So **2 outcomes** match event B.

There are **3 ways** to win coloring pencils, so **3 outcomes** match event C.

Two of the outcomes (the ball and the yo-yo) don't correspond to any of events A, B, or C.

Guided Practice

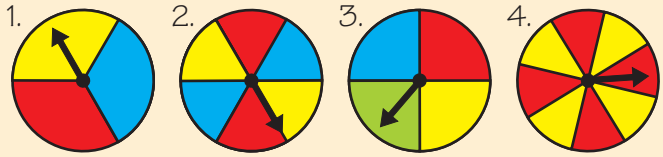
This picture shows a bag containing 5 marbles.



Exercises 1–3 are about picking marbles from the bag.
How many outcomes match the following events?

1. Picking a red marble
2. Choosing a green marble
3. Choosing a marble that is not green

Exercises 4–6 are about the four spinners shown below.



4. Which spinner has the most possible outcomes?
5. Which spinner has the least possible outcomes?
6. Latisha is looking at events that describe spinning a particular color (for example, "spinning red" or "spinning yellow").
Which spinner has the most possible events like this?
Which has the least?

An Event Can Match Very Different Outcomes

Sometimes **very different outcomes** match the **same event**.

Example 3

Eduardo rolls a normal 6-sided die. How many possible outcomes match the event "rolling an even number"?



Solution

All possible outcomes:	1	2	3	4	5	6
Possible even numbers:		**2**		**4**		**6**

There are 6 possible outcomes of rolling the die.
Only **3** of those match the event "rolling an even number."

Check it out:
The word "die" is the singular of dice.

Guided Practice

Exercises 7–12 give possible events for rolling a die.
List all the possible outcomes that match each event.

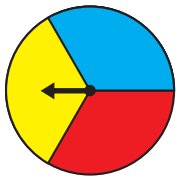
7. Event: rolling an odd number
8. Event: rolling a number less than 3
9. Event: rolling a number greater than 6
10. Event: rolling a number other than 5
11. Event: rolling a multiple of 3
12. Event: rolling a negative number

You Can Combine Outcomes of Different Actions

Events can be made up of outcomes from **different actions**.

Example 4

Kiona tosses a coin, and her friend Daniel spins the spinner shown. List all the possible outcomes of this combined action.



Solution

The outcomes are made up of the result of one toss of the coin, plus the result of one spin of the spinner.

There are **6** possible outcomes:

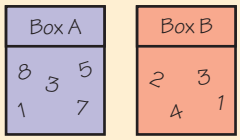
Yellow and Heads	**Blue and Heads**	**Red and Heads**
Yellow and Tails	**Blue and Tails**	**Red and Tails**

Check it out:

Once you combine outcomes of different actions, you can have many more events. Events for the combined action in Example 4 might include "heads and any color," "yellow and any result of the coin toss," or "yellow and heads." These types of events are called compound events.

Guided Practice

A two-digit number is chosen by picking one number from Box A and one number from Box B. The digit from Box A is recorded in the tens place. The digit from Box B is recorded in the ones place.



13. List all the possible two-digit numbers beginning with a 3 that could be drawn from the boxes using this method.

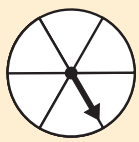
14. Which of the following is NOT a possible outcome of this method?

53 54 51 57

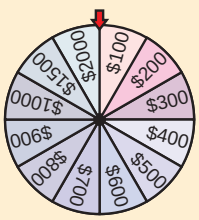
15. List all the possible outcomes for the event: "drawing a two-digit number less than 20."

Independent Practice

For each of Exercises 1–3, sketch a spinner like this one. Color or label the sections so that the spinner has exactly:



1. 2 possible outcomes matching the event "spinning blue"
2. 3 possible outcomes matching the event "spinning red"
3. 4 possible outcomes matching the event "spinning green"



This spinner is used on a game show to determine how much money a contestant wins.
Name an event using the spinner with exactly

4. 1 possible outcome.
5. 2 possible outcomes.
6. 3 possible outcomes.
7. 12 possible outcomes.

Now try these:

Lesson 6.1.1 additional questions — p457

Round Up

It's easy to list all the possible outcomes of most of the examples in this Lesson, because there aren't many to choose from. For more complicated situations, you'll need better ways to organize the information. You'll learn about some ways to do that in the next two Lessons.

Lesson 6.1.2

Tree Diagrams

California Standard:

Statistics, Data Analysis, and Probability 3.1
Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, **tree diagrams**) and express the theoretical probability of each outcome.

What it means for you:

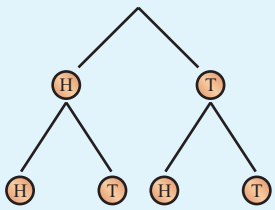
You'll learn about tree diagrams, and how you can use them to find all the possible outcomes of a situation.

Key words:

- outcome
- event
- tree diagram

Check it out:

Not all tree diagrams are drawn going across the page — sometimes the branches go down the page instead.



Check it out:

A tree diagram can show as many stages of an experiment as you want. The diagram in this example could be extended to show 10, 20, or 100 coin tosses – but it would get very large and complicated if it did.

Check it out:

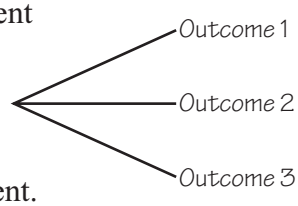
You have to read descriptions of events carefully... notice how there are 2 outcomes that match the event "a head and a tail in any order."

Making a list of possible outcomes is straightforward for simple actions like rolling dice or spinning a spinner. For more complicated situations, it can be hard to know for sure that you've listed all the possible outcomes.

A tree diagram can help you organize your work and help you check that your list is complete.

Tree Diagrams Show All Possible Outcomes

A **tree diagram** uses **branches** to show the different possible **outcomes** of each of a set of **actions** that take place.



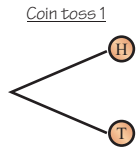
To create a tree diagram, think about the possible outcomes for each **different stage** of the experiment. Use these outcomes to build the branches.

Example 1

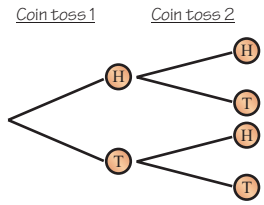
Make a tree diagram to show all the possible outcomes for the experiment of tossing a coin twice in a row.

Solution

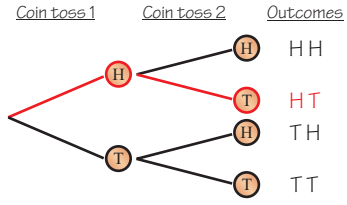
Step 1: Draw a branch for every **possible outcome** on the **first** toss of the coin.



Step 2: From **each** possible outcome of coin toss 1, draw branches for all the **possible outcomes** of coin toss 2.



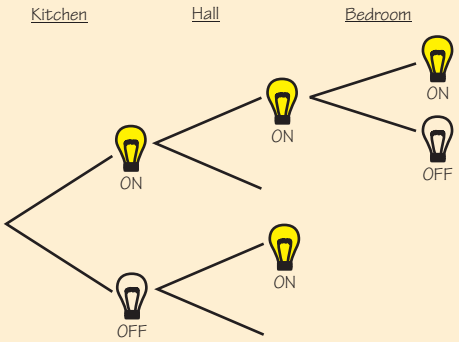
Step 3: Follow along each branch, starting at the left and finishing at the right. At the end of each branch, **list** the **outcomes** that led to that point.



From the tree, you can see that there are **4** possible outcomes.
There's 1 outcome that matches the event "**two heads**."
There's 1 outcome that matches the event "**two tails**."
There's 1 outcome that matches the event "**first a head, then a tail**,"
and 1 outcome that matches the event "**first a tail, then a head**."

Guided Practice

This tree diagram shows the ways lights in three rooms of a house can be either off or on.



1. Copy and finish the tree diagram to represent all the possible outcomes for the lights.

2. Use your tree diagram to list all the possible outcomes for the lights in the three rooms.

Use your tree diagram to list all the possible outcomes for each of the following events:

3. lights are on in exactly two of the rooms

4. lights are off in exactly two of the rooms

5. lights are on in the hall only

6. Explain why for this situation, the outcome of "on–off–off" is different from "off–off–on."

You Can Combine Any Actions Using Tree Diagrams

You can use a tree diagram to show a **combination** of different actions.

Example 2

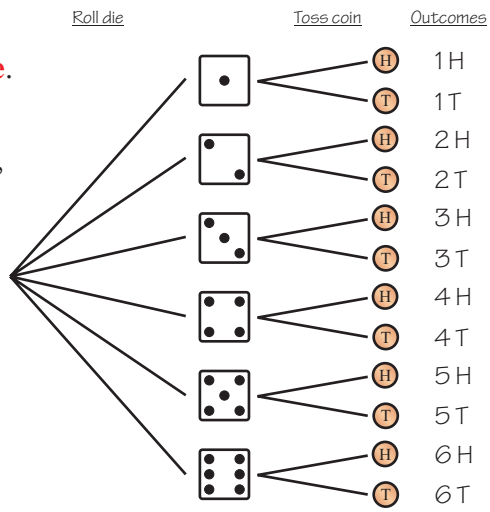
Guadalupe and Kareem are playing a game. Guadalupe rolls a die, and Kareem tosses a coin. Use a tree diagram to show all the possible outcomes of rolling one die and tossing one coin.

Solution

Step 1: Draw a branch for each possible outcome of **rolling the die**.

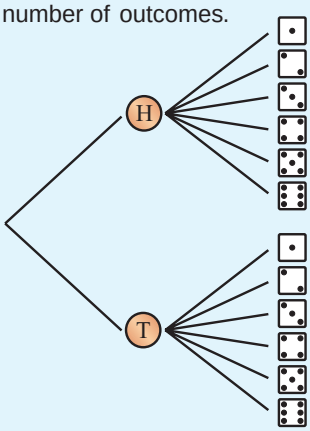
Step 2: Add branches from each possible outcome of rolling the die, showing the possible outcomes of **tossing the coin**.

Step 3: **List the outcomes** at the end of each branch.



Check it out:

You could draw the tree diagram for Example 2 by starting with the branches for tossing the coin, then adding the branches for rolling the die. You'd still get the same number of outcomes.



Guided Practice

Kim is choosing what to wear to school. She has three shirts and two pairs of pants. The shirts are white, blue, and red. The pants are black and brown.

7. Draw a tree diagram to show the different outfits Kim can wear. Begin your tree with the branches for the shirts first.

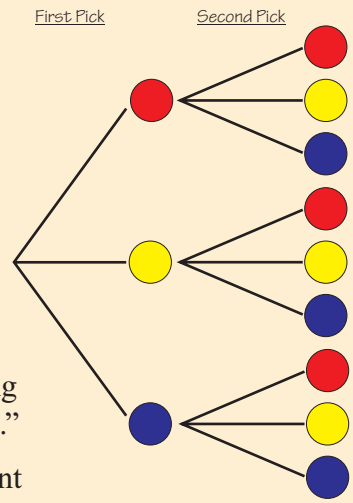
8. List all the possible outcomes. How many different outfits of one shirt and one pair of pants can Kim make?

9. Redraw the tree diagram from Exercise 7, beginning with the branches for the pants first.

10. Compare the two tree diagrams. How does changing the order affect the number of outcomes?

Independent Practice

Three different-color chips are placed in a bag. One is picked, recorded, then put back for the second pick. Use this tree diagram of the possible outcomes to do Exercises 1–5.



1. List all the possible outcomes for picking two chips. ("Blue, red" is considered different from "red, blue.")

2. List all the possible outcomes for the event "picking a second chip the same color as the first one."

3. List all the possible outcomes for "picking two chips where the first chip drawn is blue."

4. List all the possible outcomes for the event "picking two chips where a blue chip is never picked."

5. Suppose after the second draw, the color is recorded and the chip is replaced in the bag for a third draw. How many outcomes showing different orders of chips would there be? Why?

6. A bag contains 3 marbles: one purple, one yellow, and one green. In one experiment, a marble is picked and then placed back in the bag before the second marble is picked. In another experiment, a marble is picked and then kept out of the bag while the second marble is chosen. Use tree diagrams to compare the possible outcomes of drawing two marbles for each experiment.

Now try these:
Lesson 6.1.2 additional questions — p458

Round Up

Tree diagrams can be very useful for showing all the possible outcomes of a situation. But remember to draw them neatly so you can see what's going on. Make the first branches quite spread out, so that the later branches don't get all squashed up.

Lesson 6.1.3

Tables and Grids

California Standard:
Statistics, Data Analysis, and Probability 3.1
Represent all possible outcomes for compound events in an organized way (e.g., **tables**, **grids**, tree diagrams) and express the theoretical probability of each outcome.

What it means for you:
You'll see how to use a grid to find all the possible outcomes of a situation.

Key words:
- outcome
- combination
- table
- grid
- cell
- row
- column

Check it out:
The individual boxes in a grid or table are called cells.

Check it out:
When you fill in a table with letter codes like this, the first letter represents the first action (the first toss of the coin in this case), and the second letter comes from the second action.

Last Lesson you learned how to use tree diagrams to help you organize your information about the possible outcomes of a situation. In this Lesson, you'll see some other ways to organize this type of information.

Tables Can Help You to Organize Your Information

Tables and **grids** use **rows** and **columns** to show all the possible outcomes for a **combination** of **two** actions.

Example 1

Use a table to find the possible outcomes for tossing a coin twice.

Solution

Step 1: Write the possible outcomes for the **first** toss of the coin in a **row** along the top of your table.

Step 2: Write the possible outcomes for the **second** toss of the coin in a **column** down the side of your table.

1st coin toss

2nd coin toss	Heads	Tails
Heads		
Tails		

Step 3: Use the rows and columns to make a **grid** of cells.

Step 4: Fill in each cell of the grid by **combining** the outcomes listed on the row and column of that cell.

1st coin toss

2nd coin toss	Heads	Tails
Heads	H H	T H
Tails	H T	T T

This cell represents getting Tails on the first toss and Heads on the second toss, so the cell contains TH.

The grid shows the **4** possible outcomes for tossing a coin twice: **HH**, **HT**, **TH**, **TT**.

Guided Practice

1. A family is chosen at random from all the families in California that have 2 children.

Copy and complete the table to show all the possible combinations of boys and girls the family could have.

1st child

2nd child	Boy	Girl
Boy		
Girl		

The cells in the table below represent the outcomes for drawing a marble from a bag, recording it, and drawing another marble.

2nd pick \ 1st pick	?	?	?
Green	WG	GG	BG
?	WW	GW	BW
?	WB	GB	BB

Key: B = blue G = green W = white

2. Copy the table and fill in the color of marble that belongs in place of each question mark.

3. How many possible outcomes match the event "picking at least one blue marble"?

4. How many possible outcomes match the event "not picking any green marbles"?

Tables Are Useful in Situations Involving Numbers

It can be especially helpful to use a table when you need to do **calculations** with numbers in the rows and columns of the table.

Example 2

Two dice are rolled. Find all the possible outcomes that match the event "a sum of 7 is rolled."

Solution

Step 1: Find the possible outcomes for the first die: **1, 2, 3, 4, 5, or 6**. Write these in a **row** along the top of your table.

Step 2: Find the possible outcomes for the second die: **1, 2, 3, 4, 5, or 6**. Write these in a **column** down the side of your table.

Step 3: Create a **grid** of cells inside the table.

Step 4: Fill in each cell of the grid by **adding** the outcomes listed on that row and column.

2nd die rolled \ 1st die rolled	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

In this case, the grid shows there are **6 possible outcomes** that have a sum of 7.

You can look at the column and row of each of these 6 cells and write the outcomes with a sum of 7: **(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)**.

(1, 6) means 1 on the first roll, 6 on the second roll.

Guided Practice

Exercises 5–7 are about outcomes for rolling two dice and adding the results together. Use the grid from Example 2 to find the answers.

5. List all the possible outcomes with a sum of 4.
6. How many possible outcomes are there for the event "a sum of 6"?
7. Describe an event that has only one possible outcome.

Two dice are rolled. The number on the second die is subtracted from the number on the first die. The result is called n.

8. Draw a grid showing all the possible outcomes of this experiment.
9. How many possible outcomes match the event "$n = 2$"?
10. How many possible outcomes match the event "n is negative"?

Independent Practice

1. An ice-cream truck sells 4 flavors of ice cream: vanilla, chocolate, mint, and strawberry; and 3 ice cream toppings: marshmallows, sprinkles, and whipped cream. Create a table of the possible outcomes for choosing 1 ice cream flavor with 1 ice cream topping.

Exercises 2–3 are about a computer game that allows you to choose the hair and eye color of a character.
Hair color choices are: brown, black, and blond.
Eye color choices are: brown, green, and blue.

2. Draw a table that shows all the possible hair and eye color outcomes. (Tip: use Br, Bk, and Bd for the hair color abbreviations, and Br, G, and Bl for eye color.)

3. How would adding red hair and hazel eyes to the choices affect the number of possible outcomes?

Exercises 4–9 are about a math game. Two dice are rolled and the numbers on the dice are multiplied.

4. Create a table to show all the possible products of the two dice.

Use your table from Exercise 4 to find all the possible outcomes matching each of the following events:
5. A product of 12
6. A product of 4
7. A product greater than 20

Use your table from Exercise 4 to name the following:
8. An event that has exactly 1 outcome.
9. An event that has at least 20 outcomes.

Now try these:
Lesson 6.1.3 additional questions — p458

Round Up

Tables and grids are most useful for combining two actions that both have a few possible outcomes. A grid can show that sort of situation much more clearly than a tree diagram with lots of branches. But the problem with grids is that they can't show combinations of more than two actions.

Section 6.2 introduction — an exploration into:
Heads or Tails

If you were to toss a coin, what's the probability that it will land on heads? The answer is 0.5 — but what does that actually mean? That's what this Exploration is all about.

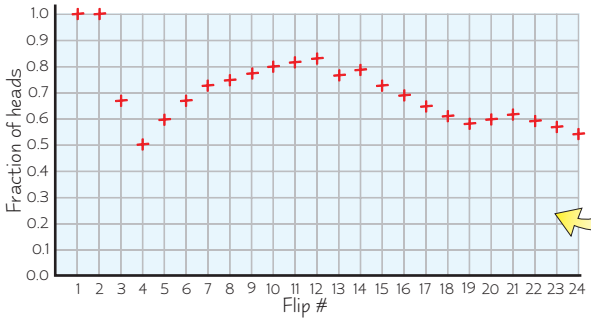
If you flip a coin **24 times**, how many times do you predict it will land on **heads**?
Write down your prediction. You're now going to try it for real...

Flip a coin **24 times**. Record your **results** in a chart like this one.

After each flip, calculate the fraction of your total flips so far that landed on heads. Write your result in the last column.

Flip #	H or T?	Fraction of heads
1	H	1/1 =1.0
2	H	2/2 =1.0
3	T	2/3 =0.67
4	T	2/4 =0.5
5	H	3/5 =0.6
6	H	4/6 =0.67
7	H	5/7 =0.71

Divide the number of heads by the total flips so far.
Then convert to a decimal using a calculator.



Show your information on a graph like this one.
- Plot the **flip number** along the horizontal axis.
- Plot the **fraction of heads** on the vertical axis.

Example

1. Use the above graph to say what fraction of the flips were heads after: **a.** 8 flips, **b.** 24 flips.
2. What happens to the line on the graph as you move to the right?

Solution

1. The fraction of heads was: **a.** **0.75 after 8 flips**, **b.** **0.53 after 24 flips**.

2. The graph seems to be **approaching 0.5**, and is **zig-zagging a lot less** than at the start.

Exercises

1. Compare your graph to another student's. Describe any similarities and differences.
2. After 24 flips, what fraction of your flips were heads?
 Find the difference between this fraction and 0.5.
3. If you flipped the coin 50 times, do you think your answer to Exercise 2 would change?
 Explain your answer using your graph and what you know about probability.
4. Now combine your data with the data from the rest of the class.
 Find: (i) the total number of times a coin was flipped, and (ii) the total number of heads.
 Find the fraction of heads, and the difference between this fraction and 0.5.
 How does this compare with your answer to Exercise 2?
5. What would you expect if you flipped a coin 1000 times? 10,000 times?

Round Up

For the first few flips, your graph may have zig-zagged up and down a bit. But as you did more flips, things should have settled down somewhere near 0.5, since that's the theoretical probability. The idea of what probability actually means is tricky, but you'll see more about it in this Section.

Lesson 6.2.1

Section 6.2 Probability

A lot of the time, you can't say for sure whether or not one particular event will happen. But you can often say how good the chances are. Probability is a way of using numbers to describe the chance of an event happening.

California Standard:

Statistics, Data Analysis, and Probability 3.3

Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, $1–P$ is the probability of an event not occurring.

What it means for you:

You'll learn about using probability as a way to describe how likely events are to happen.

Key words:

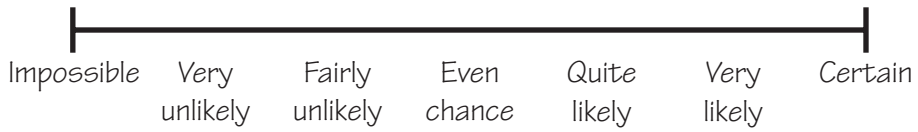
- probability
- chance
- likely
- percent
- fraction
- decimal

Some Events Are More Likely to Happen Than Others

People often talk about things that **might happen**, using words like "**chance**," "**likely**," and "**probability**":

"What is the **probability** it will snow today?"
"How **likely** is it that the school football team will win its next game?"
"What is the **chance** that you will go to a movie this weekend?"

Look at the line below and think about where your answer would be for each of these questions.



Guided Practice

Decide where you would put the chances of the following events happening on the scale above.

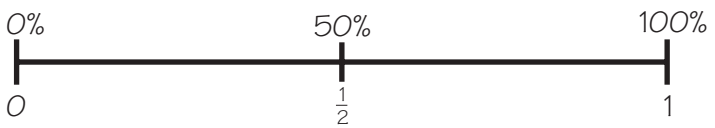
1. Leaves falling from the trees next fall.
2. Finding a live elephant in your bedroom when you get home.
3. Winning a raffle if you have 1 out of 100 tickets.
4. Winning a raffle if you have 99 out of 100 tickets.
5. Winning a raffle if you have 1 out of 1,000,000 tickets.
6. A tossed coin landing on heads.

7. Put the events named in Exercises 1–6 in order, from most likely to least likely.

Probability Is a Way to Say How Likely an Event Is

In math, **probability** is a way of describing the chance that an event will occur. Probability can be written using **fractions**, **decimals**, or **percents**.

You can replace the words on the line above with **numbers** that represent how likely an event is to occur.



Check it out:

The closer the probability gets to 1, the more likely the event is to happen.
The closer the probability is to 0, the less likely the event is.

Check it out:
If you did a very large number of experiments with 2 outcomes that each have a probability of 50% (tossing a coin, for example), then you would expect that about half the experiments would have one outcome (heads, say), and about half the experiments would have the other outcome (tails).

A probability of **0** (or **0%**) means that there is **no chance**.

A probability of **1** (or **100%**) means that the event will **definitely** happen.

A probability of $\frac{1}{2}$ (or **50%**) means that the event **might happen**, but there's an **equal chance** that it **won't**.

Guided Practice

Estimate the probability that each of the following things will happen. Write your answers as percents.

8. It will go dark tonight.

9. Your math teacher will turn into a pineapple.

10. A 6th-grader from California chosen at random will be a girl.

Use the list of probabilities below to answer Exercises 11–12.

$$\frac{1}{2} \qquad \frac{5}{8} \qquad \frac{0}{25} \qquad \frac{10}{10}$$

11. Which of the probabilities above represents an impossible event?

12. Which of the probabilities above represents a certain event?

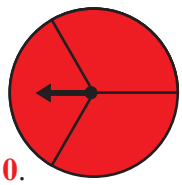
Don't forget:
Calculating an exact probability isn't the same as saying whether an event will definitely (or definitely not) happen. See Lesson 6.4.2 for examples of where you can't find an exact probability.

Probability Is Usually Calculated Exactly

There are many situations where you can say **exactly** what the probability of an event is.

Example 1

What is the probability of spinning the color red on this spinner? What is the probability of spinning the color blue?



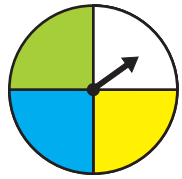
Solution

Spinning the color red is **certain**, so the probability is **1**.

Spinning the color blue is **impossible**, so the probability is **0**.

Example 2

What is the probability of spinning the color blue on this spinner?



Solution

The blue section is **one-fourth** of the spinner.

So the probability of spinning the color blue is

$$\frac{1}{4} = 0.25 = 25\%$$

If you spun the spinner many times, about one-fourth of the spins would land on blue.

Check it out:
By looking at the spinner, you can see that there's a 1 in 4 chance of spinning the color blue. A chance of 1 in 4 is written mathematically as a probability of $\frac{1}{4} = 0.25 = 25\%$.

Guided Practice

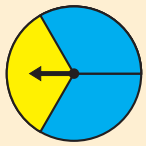
Exercises 13–15 are about the spinner shown.
Find the probability of spinning the colors below.
Write your answers as decimals.



13. Blue **14.** Yellow **15.** Pink

In Exercises 16–18, find the probability of spinning the color yellow on each of the following spinners. Write your answers as fractions.

16. 

17. 

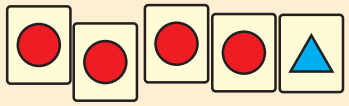
18. 

Independent Practice

Each set of cards shown below is turned over and shuffled, then one card is picked. For each set, find the probability of picking

1. a triangle card **2.** a star card

A. 

B. 

C. 

D. 

A bag has 1 red, 1 blue, and 2 yellow marbles in it.



3. How many marbles are in the bag?
4. How many marbles are red?
What is the probability of drawing a red marble?
5. How many marbles are blue?
What is the probability of drawing a blue marble?
6. How many marbles are yellow?
What is the probability of drawing a yellow marble?

7. The probability of an event occurring is $\frac{3}{8}$.

Which two values below represent this same probability?

0.375 375% 0.375% 37.5% $\frac{3}{8}\%$

8. The probability of an event occurring is 55%.
Which two values below represent this same probability?

55 5.5 $\frac{0.55}{100}$ $\frac{55}{100}$ $\frac{11}{20}$

9. In a tiled hallway, kids are jumping from one tile to the next.
The probability of landing on a green tile is 60%.
What fraction of the hallway area is covered with green tiles?

Don't forget:
For more information about fractions, percents, and decimals, see Chapter 3.

Now try these:
Lesson 6.2.1 additional questions — p458

Round Up

Probability is useful because you can use it to compare the chances of different events happening. The event with the highest probability is the most likely to occur.

Lesson 6.2.2

Expressing Probability

In this Lesson, you'll learn how to find the probability of an event by looking at all the possible outcomes.
There's a handy formula you can use to do this.

California Standard:

Statistics, Data Analysis, and Probability 3.3

Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, $1–P$ is the probability of an event not occurring.

What it means for you:

You'll meet and use an important formula that will help you to work out probabilities.

Key words:

- probability
- event
- outcome
- favorable

Check it out:

Another way to think of probability is as a ratio. The probability of an event is the ratio of favorable outcomes to possible outcomes.

Check it out:

This formula only works if all the possible outcomes are equally likely. For the actions in this Section, this is the case (for example, when you roll a die, all the different numbers are equally likely).

Don't forget:

It's usually a good idea to write fractions in their simplest form.
For example, a probability of $\frac{40}{100}$ can be written as $\frac{2}{5}$ — see Lesson 3.4.2 for more information.

You Can Calculate Probabilities by Counting Outcomes

There's some special **probability notation** you'll need to know.

"**P(A)**" means "**the probability that event A will happen**."

So for tossing a coin, you can write "the probability of getting heads is $\frac{1}{2}$" as "P(Heads) = $\frac{1}{2}$."

You can often find probabilities by thinking about the **possible outcomes** of the situation.

$$\text{P(event)} = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

Favorable outcomes are the outcomes that match the event.

Example 1

Find the probability of rolling a 2 on a standard 6-sided die.

Solution

The possible outcomes are: 1 2 3 4 5 6

The favorable outcomes are: 2

There are **6 possible outcomes**, and **1 favorable outcome**.

So $P(2) = \frac{1}{6}$

Guided Practice

Write the answers to Exercises 1–7 as fractions.

Jaden has a bag containing 100 marbles.
There are 50 red marbles, 30 blue marbles, and 20 green marbles.
He picks out one marble. Find the probability that the marble is:

1. Red **2.** Blue **3.** Green

There are 50 socks in Jasmine's drawer. There are 25 black socks, 10 blue socks, 10 orange socks, and 5 striped socks. Jasmine picks a sock without looking. Find the probability that the sock is:

4. Black **5.** Blue **6.** Orange **7.** Striped

Read the Description of an Event Very Carefully

Always read the situation carefully. And then think very carefully too.

Don't forget:
A standard pack contains 52 cards. There are 4 suits: hearts, clubs, spades, and diamonds. Hearts and diamonds are red, while clubs and spades are black.
Each suit has 13 cards: the numbers from 1–10 (1 is usually called an "ace"), plus the jack, queen, and king (these are called "face cards").

Example 2

What is the probability of picking a queen out of a standard pack of 52 playing cards?

Solution

There are **52 possible outcomes**. The pack of cards contains **4 queens**.

So $P(\text{queen}) = \frac{4}{52} = \frac{1}{13}$

Guided Practice

Find the probability of the following events when rolling a die once. Give your answers as fractions in their simplest form.

8. Rolling an odd number
9. Rolling an even number
10. Rolling a number more than 2
11. Rolling a prime number

Find the probability of the following events when picking one card from a standard pack of 52.

12. Picking the jack of diamonds
13. Picking a seven
14. Picking a club
15. Picking a black card

Don't forget:
A prime number can't be divided evenly by any other numbers except itself and 1. (The number 1 isn't prime.)

Probability Can Give You Information About Outcomes

If you know the **probability** of an event, you might be able to figure out numbers of **outcomes**.

Example 3

The probability of picking a red marble out of a bag is 25%.
There are 20 marbles in the bag. How many red marbles are there?

Solution

$$P(\text{red}) = 25\% = \frac{1}{4} = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

$$\frac{1}{4} = \frac{\text{Number of favorable outcomes}}{20}$$ There are 20 marbles in total

$$\text{Number of favorable outcomes} = 20 \times \frac{1}{4}$$ Multiply both sides by 20

$$= 5$$

So there are **5 red marbles**.

Guided Practice

Exercises 16–21 are about picking one marble (without looking) from a bag full of marbles. Each exercise is about a different bag.

16. P(picking yellow) = $\frac{1}{2}$, and there are 11 yellow marbles. How many marbles are there altogether?

17. If there are 100 marbles, and P(picking red) = 10%, how many red marbles are there?

18. P(picking blue) = 25%, and there are 8 blue marbles. How many marbles are there in total?

19. P(picking black) = $\frac{1}{3}$. If there are 10 black marbles, how many marbles are there in total?

20. There are 80 marbles in the bag in total. P(picking silver) = $\frac{3}{4}$. How many silver marbles are in the bag?

21. There are 5 green marbles in the bag. What is the total number of marbles if P(picking green) = 20%?

Independent Practice

1. The winning contestant on a game show picks one of 80 boxes to find out what prize they win.

1 box contains $1,000,000
4 boxes contain vacation tickets
10 boxes contain cameras
15 boxes contain $10
20 boxes contain a pair of socks
30 boxes contain signed photos of the game show host

Find the probability of winning each prize as a fraction.

Sixty-four raffle tickets have been sold.
Maria bought 5 of the tickets and her friend Kyle bought 10.
Write the probabilities of the following events in fraction form:

2. Maria will win the raffle.
3. Kyle will win the raffle.
4. Someone other than Maria or Kyle will win the raffle.
5. If Maria wanted to have a 25% chance of winning, how many of the 64 tickets would she need to purchase?

Destiny buys 20 tickets, so that 84 tickets have now been sold.

6. How does this affect Kyle's and Maria's chances of winning? Use percents rounded to the nearest whole percent to compare the probability of winning before and after Destiny's purchase.

Now try these:
Lesson 6.2.2 additional questions — p459

Round Up

Counting outcomes is fairly straightforward in simple situations like these.
But for more complicated sets of events and outcomes, you'll need to organize your information.

Lesson 6.2.3

Counting Outcomes

Last Lesson, you saw how to find probabilities with a formula. But to use it, you need to know how many possible and favorable outcomes there are. Luckily, you learned two ways to do that a few Lessons ago. Now you'll see just how useful tables, grids, and tree diagrams can be.

Tables and Grids Help to Calculate Probabilities

You know that: $$P(\text{event}) = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

So if you can make an **organized list** of possible outcomes, it makes it easier to find probabilities.

Example 1

Two dice are rolled. Find the probability that the sum of the numbers rolled is: (i) 8, and (ii) greater than 8.

Solution

(i) This table shows all the possible outcomes for the sum of the numbers rolled on two dice.

All you need to do is count how many possible and favorable outcomes there are, and put them into the formula.

There are **5 ways to roll 8**, and there are **36 possible outcomes** altogether.

2nd die rolled \ 1st die rolled	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

So, $P(\text{rolling 8}) = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}} = \frac{5}{36}$

(ii) The favorable outcomes for the event "rolling a sum greater than 8" are the outcomes 9, 10, 11, and 12. In total, this makes **10 outcomes**.

So, $P(\text{rolling over 8}) = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}} = \frac{10}{36} = \frac{5}{18}$

Guided Practice

Use the table above to find the probability of each of the following events when rolling two dice:

1. Rolling a sum of 9
2. Rolling a sum of 10
3. Rolling a sum of 2
4. Rolling a sum less than 6
5. Rolling a sum of 17
6. Rolling a sum that is a prime number

California Standards:

Statistics, Data Analysis, and Probability 3.1

Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.

Statistics, Data Analysis, and Probability 3.3

Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, $1–P$ is the probability of an event not occurring.

What it means for you:

You'll see how ways of counting outcomes such as grids and tree diagrams can help you work out probabilities of events.

Key words:

- probability
- outcome
- event
- grid
- table
- tree diagram

Don't forget:

The table used here is the same as the one from Example 2 of Lesson 6.1.3.

Don't forget:

To find the probability of rolling a sum less than 6, count how many outcomes of 2, 3, 4, and 5 there are.

Check it out:
The easiest way to tackle Exercises 7–16 is to start each answer by writing a fraction of the form $\frac{?}{300}$. Then you can convert to percents by multiplying by 100.

The table below lists the number of students at a middle school by grade and gender.

One student is chosen at random. Determine the probability of choosing each type of student listed in Exercises 7–16. Write each probability as a percent.

	Boys	Girls	Totals
Grade 6	57	51	108
Grade 7	42	57	99
Grade 8	45	48	93
Totals	144	156	300

7. P(girl)
8. P(boy)
9. P(6th grader)
10. P(7th grader)
11. P(8th grader)
12. P(6th grade boy)
13. P(7th grade girl)
14. P(8th grade boy)
15. P(9th grader)
16. P(6th, 7th, or 8th grade student)

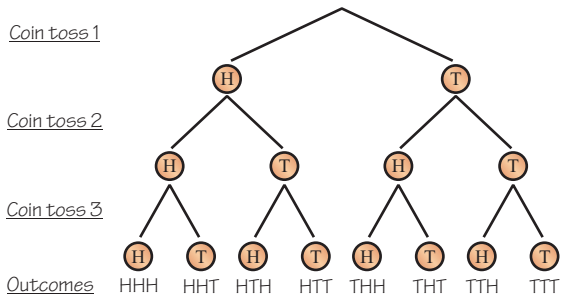
Tree Diagrams Can Show You All Possible Outcomes

Tree diagrams can help you find probabilities too.

Example 2

A coin is tossed three times.
Find the probability of getting tails on exactly two of the three tosses.

Solution



This tree diagram shows all the possible outcomes for tossing a coin three times.

There are **8 possible outcomes**.

There are **3 outcomes** that have exactly two tails: HTT, THT, and TTH.

So $\text{P(exactly 2 tails)} = \dfrac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}} = \dfrac{3}{8}$

Guided Practice

Write your answers to Exercises 17–21 as decimals.
Use the tree diagram above to find the probability of each of the following events for tossing a coin three times:

17. Heads on exactly 2 tosses
18. Tails on 2 or more tosses
19. Tails on fewer than 2 tosses
20. The coin landing on the same side all 3 tosses
21. The first and third coins landing on the same side

Independent Practice

Exercises 1–12 are about the color spinner shown below.



The spinner is spun once.
Find the following probabilities.

1. P(green) **2.** P(yellow)
3. P(white) **4.** P(not purple)

5. The spinner is spun twice. Make a tree diagram or a table that shows all the possible outcomes. Use the abbreviations R for red, Y for yellow, G for green, and B for blue.

Use your diagram from Exercise 5 to answer Exercises 6–12.
6. How many possible outcomes are there?
7. How many ways can you get 2 spins of the same color?

Find the following probabilities.
Write your answers as fractions in their lowest terms.
8. P(2 spins the same color)
9. P(one blue and one green in any order)
10. P(blue first, then green)
11. P(exactly one spin is red)
12. P(at least one spin is red)

Don't forget:
The product of two numbers is the result of multiplying them together.

Contestants on a game show spin a wheel twice.
If the product of the numbers is odd, they win a prize.
The numbers on the wheel are 1, 2, 4, 5, and 6.

13. Make a table with a grid that shows all the possible products from any two spins of the wheel.
14. What is the probability that a contestant will win?
15. What is the probability that a contestant will lose?

The wheel in Exercises 13–15 had two odd numbers and three even numbers on it.

16. Design a new wheel that has an equal number of odd and even numbers on it.
17. Use your new wheel. Determine which probability is greater, P(even product) or P(odd product).
18. If your wheel in Exercise 16 had a different number of integers on it (but half of those integers were still even, with the rest being odd), would your answer to Exercise 17 have been different? Explain your answer.

Now try these:
Lesson 6.2.3 additional questions — p459

Round Up

The most important use of tree diagrams and tables of outcomes is in finding probabilities. Look back at Lessons 6.1.1–6.1.3 if you need a reminder about anything to do with listing possible outcomes.

Lesson 6.2.4

Probability of an Event Not Happening

California Standards:

Statistics, Data Analysis, and Probability 3.3

Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, $1-P$ is the probability of an event not occurring.

What it means for you:

You'll learn how to find the probability that a particular event does not happen.

Key words:

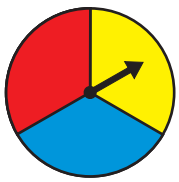
- probability
- event
- outcome
- favorable

The probabilities you've seen so far represent the chances that an event will happen. In this Lesson, you'll learn how to find the probability that an event doesn't happen. You'll be pleased to know that the math you'll need to do isn't much different from what you've been doing so far in this Section.

Find Probabilities by Counting Outcomes

So far you have worked out **probabilities** of events **happening**. You can also find the probability that an event **doesn't happen** by counting the number of outcomes that **don't match** the event.

Example 1



Ruben spins this spinner.

Find the probability that Ruben spins the color yellow. What is the probability that Ruben does not spin the color yellow?

Solution

There are **3 possible outcomes**: red, yellow, and blue.

So $P(\text{Yellow}) = \frac{1}{3}$

If he doesn't spin yellow, Ruben must spin either **red or blue**. So there are **2 favorable outcomes** for **not** spinning yellow.

So $P(\text{Not yellow}) = P(\text{Red or blue}) = \frac{2}{3}$

Guided Practice

In Exercises 1–8, determine each probability for one spin of the spinner shown below. Give your answers as simplified fractions.



1. P(Green)
2. P(Not green)
3. P(Yellow)
4. P(Not yellow)
5. P(Red)
6. P(Not red)
7. P(Orange)
8. P(Not orange)

Check it out:

In some situations, P(not event A) can also be written as P(event B). For example, for tossing a coin, P(not tails) is the same as saying P(heads).

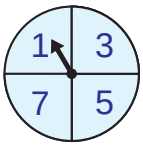
You Can Find P(not A) by Counting Outcomes

You can use the probability that event A **will** happen to find the probability that A **won't** happen.

Check it out:
Two events are said to be complementary if between them they cover all the possible outcomes and if only one of them can happen. So "A" and "not A" are complementary (since an event has to either happen or not happen). "Not A" is called the complement of "A."

Example 2

Mario is playing a game using the spinner shown below.
He spins it twice, and adds the numbers he spins to get his score.



This table gives the possible outcomes.

If A is the event "Mario scores 8," then find: (i) P(A), (ii) P(not A).

1st spin \ 2nd spin	1	3	5	7
1	2	4	6	8
3	4	6	8	10
5	6	8	10	12
7	8	10	12	14

What do you notice about P(A) + P(not A)?

Solution

(i) The table shows there are 16 possible outcomes.
And there are 4 possible outcomes where Mario does score 8.

So P(Mario scores 8) $= \frac{4}{16} = \frac{1}{4}$ (or 25%).

(ii) But if there are 4 possible outcomes where Mario **does** score 8, then there must be 16 – 4 = **12** outcomes where Mario **does not** score 8.

So P(Mario does not score 8) $= \frac{12}{16} = \frac{3}{4}$ (or 75%).

Notice that **P(A) + P(not A) = 1** (or 100%).

Check it out:
If event A is "Mario scores 8," then event not A is "Mario does not score 8."

You Can Also Find P(not A) If You Know P(A)

The result about P(A) and P(not A) adding up to 1 is always true.

> P(event A happening) + P(event A not happening) = 1 = 100%

Check it out:
You can write this rule as: P(A) + P(not A) = 1.

Example 3

The weather channel says there is a 30% chance of rain today.
What is the probability that it will not rain today?

Solution

P(rain) + P(not rain) = 100%
so 30% + P(not rain) = 100%

This means P(not rain) = 100% – 30% = **70%**

Guided Practice

Exercises 9–16 give P(A). Find P(not A) in each case.

9. P(A) = $\frac{3}{4}$ **10.** P(A) = 0.58 **11.** P(A) = 11% **12.** P(A) = 43%

13. P(A) = 0.9 **14.** P(A) = $\frac{11}{12}$ **15.** P(A) = 1 **16.** P(A) = 0

Guided Practice

Find the following probabilities for Mario's game from Example 2.
Write your answers as decimals

17. P(Mario scores 14)
18. P(Mario doesn't score 14)
19. P(Mario scores 6)
20. P(Mario doesn't score 6)
21. P(Mario scores less than 10)
22. P(Mario doesn't score less than 10)
23. P(Mario scores 7)
24. P(Mario doesn't score 7)

25. Jenna rolls a die. Event A is "rolling a number greater than 4."
Event B is "rolling a number less than 4."
Jenna claims that P(event A) + P(event B) = 1.
Explain to Jenna why this is not true.

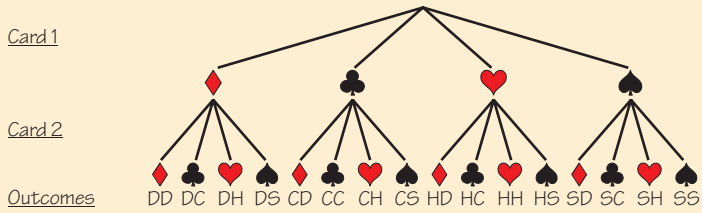
Independent Practice

1. The probability of an event occurring is 0.6. Explain why the probability of the event not occurring cannot be 0.2.

2. P(A occurring) and P(A not occurring) are the same. Find P(A).

3. P(A occurring) is twice P(A not occurring). Find P(A).

Cynthia picks one card from a standard pack of 52 cards.
She makes a note of the suit, then replaces the card and picks another one. The tree diagram below shows the possible outcomes.



D, C, H, and S represent diamonds, clubs, hearts, and spades.
Calculate P(A) and P(not A), where event A is Cynthia picking the following:

4. At least one heart
5. Two diamonds
6. At least one red card
7. One heart and one club
8. Two cards the same suit

Now try these:
Lesson 6.2.4 additional questions — p459

Round Up

The rule P(A) + P(not A) = 1 is very useful and important. If you need to find a complicated-looking P(not A), the first thing you should think is "Is it easier to find 1 – P(A)?"
The answer is quite often yes.

Lesson 6.2.5

Venn Diagrams

California Standard:

Statistics, Data Analysis, and Probability 3.1

Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.

What it means for you:

You'll learn about Venn diagrams, which are useful in helping you to understand how different events relate to each other.

Key words:

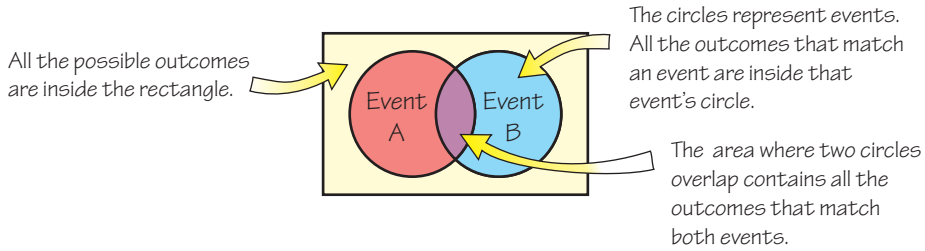
- Venn Diagram
- outcome
- event

It's often tricky to figure out in your head how different events and outcomes are related. A Venn diagram is a way to show how different events are related, and they can make probabilities easier to visualize.

A Venn Diagram Is a Way to Represent Events

One **outcome** will often match more than one **event**.

You can show situations where **one or more outcomes** match **more than one event** using a **Venn diagram**.



The next example should make the usefulness of a Venn diagram clearer.

Example 1

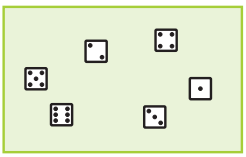
The following are two events for rolling a die once:
Event A: Rolling an even number
Event B: Rolling a number less than 4

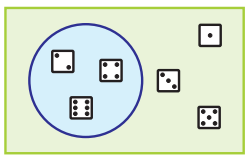
Use a Venn diagram to show how many outcomes match both events.

Solution

The rectangle represents **all possible outcomes**.

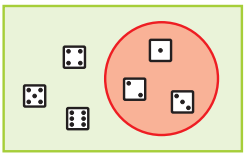
This means rolling 1, 2, 3, 4, 5, or 6.

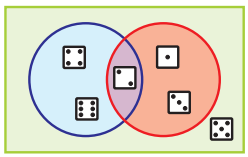




The blue circle represents **event A**, rolling an even number.

The red circle represents **event B**, rolling a number less than 4.





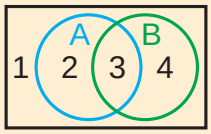
The outcome "**rolling a 2**" is in both circles. The circles have to **overlap**, so that 2 can be in both at the same time.

There is **1 outcome** that matches both event A and event B.
There is also **1 outcome** (5) that matches neither event A nor event B.

Check it out:
Many Venn Diagrams don't show all the actual outcomes on the diagram. They often just show the circles representing the events.

Guided Practice

Andres picks a card from a standard pack.
Event A is "picking a spade." Event B is "picking an ace."



In which section of this Venn diagram do the following outcomes belong?

1. Ace of clubs **2.** King of hearts
3. Ace of spades **4.** Three of spades

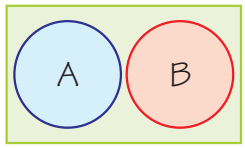
5. Sketch a Venn diagram showing the events below if an integer from 1 to 25 is picked at random. Place the integers 1 to 25 in the correct areas of the Venn diagram.

Event A: the number picked is a multiple of 4
Event B: the number picked is a multiple of 6

The Circles on a Venn Diagram Don't Always Overlap

Venn diagrams can show some other situations:

The circles **don't overlap** at all if **no outcomes** match both event A and event B.

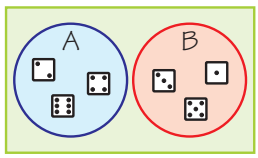


Example 2

Draw a Venn diagram showing the following events for rolling one die:

Event A: Rolling an even number Event B: Rolling an odd number

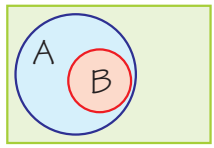
Solution



The outcomes 2, 4, and 6 match event A.
The outcomes 1, 3, and 5 match event B.

No outcomes match both events, so the circles **don't overlap**.

In this diagram, all the outcomes matching **event B** also match **event A**. Some outcomes match A, but not B.

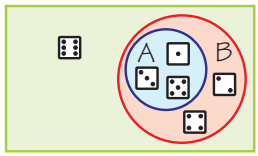


Example 3

Draw a Venn diagram showing the following events for rolling one die:

Event A: Rolling an odd number Event B: Rolling less than 6

Solution

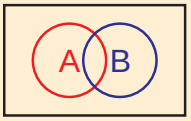


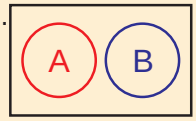
The outcomes 1, 3, and 5 match event A.
The outcomes 1, 2, 3, 4, and 5 match event B.

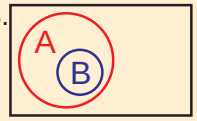
All the outcomes matching event A also match event B. The circle representing event A is **completely inside** the one for event B.

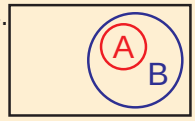
Guided Practice

Use the Venn diagrams below to answer Exercises 6–11

1. 

2. 

3. 

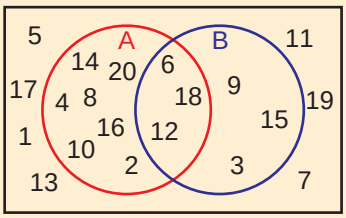
4. 

Which diagram could show each of the following pairs of events for picking a number at random from 1 to 100?

6. Event A: odd — Event B: even
7. Event A: less than 50 — Event B: even
8. Event A: less than 50 — Event B: less than 20
9. Event A: greater than 39 — Event B: less than 74
10. Event A: greater than 86 — Event B: less than 17
11. Event A: a multiple of 6 — Event B: a multiple of 3

Independent Practice

This Venn diagram shows two events when a number from 1 through 20 is chosen at random.
Use it to answer Exercises 1–6.



Find how many outcomes match

1. event A **2.** event B
3. both event A and event B
4. at least one of events A and B

5. Which of the following could be event A:
A. multiple of 4 B. number less than 16 C. even number

6. Which of the following could be event B:
A. multiple of 3 B. number less than 18 C. odd number

Sketch Venn diagrams for the following pairs of events. Place the integers from 1 to 12 in the correct areas of the Venn Diagram.

7. Event A: choosing an odd number
Event B: choosing a multiple of 4

8. Event C: choosing a number less than 6
Event D: choosing a prime number

9. Event E: choosing a number greater than 4
Event F: choosing a multiple of 5

Now try these:
Lesson 6.2.5 additional questions — p460

Round Up

Venn diagrams often don't give enough information on their own to figure out probabilities, but they can still be useful. In the next Lesson, you'll see that when you combine events, a Venn diagram can help you to understand the situation better.

Lesson 6.2.6

Combining Events

The words "and" and "or" are simple little words people use all the time, but in probability they're very important. In this Lesson, you'll see how they describe different probabilities when two events are combined.

"And" Means "Both Events Happen"

Some probabilities **combine two events**.

P(A **and** B) means "the probability that event A and event B **both** happen."

Example 1

This spinner is spun once.



Event A is spinning an even number
Event B is spinning a number less than 7.
Find P(event A and event B)

Solution

Possible outcomes	1	2	3	4	5	6	7	8	9	10
Event A		2		4		6		8		10
Event B	1	2	3	4	5	6				
Event A and Event B		2		4		6				

There are **10 possible outcomes**.
There are **3 outcomes** that match both events A and B.

So P(A and B) = P(even number and less than 7) = $\frac{3}{10}$ = **30%**

Guided Practice

Exercises 1–4 use the spinner from Example 1. For one spin of the spinner, find P(event A and event B) for the following pairs of events:

1. Event A: Spinning an odd number
 Event B: Spinning a number greater than 7
2. Event A: Spinning a prime number
 Event B: Spinning a number less than 4
3. Event A: Spinning a number greater than 2
 Event B: Spinning a multiple of 3
4. Event A: Spinning an odd number
 Event B: Spinning an even number

California Standards:

Statistics, Data Analysis, and Probability 3.3

Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if *P* is the probability of an event, 1–*P* is the probability of an event not occurring.

Statistics, Data Analysis, and Probability 3.4

Understand that the probability of either of two disjoint events occurring is the sum of the two individual probabilities and that the probability of one event following another, in independent trials, is the product of the two probabilities.

What it means for you:

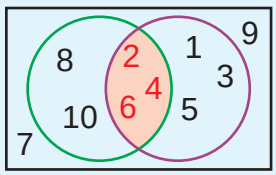
You'll work out the probability of combinations of events.

Key words:

- probability
- event
- outcome
- and
- or

Check it out:

You can use a Venn diagram to help you understand what's going on in Example 1.



Event A is the green circle, event B is the purple circle. The outcomes that match both events are in the area where the circles cross.

"Or" Means "At Least One Event Happens"

Check it out:
P(A or B) means the probability that either A happens, B happens, or they both happen.

P(A **or** B) means "the probability that **at least one of** event A and event B happens."

Example 2

This spinner is spun once.



Event A is spinning an even number
Event B is spinning a number less than 7.
Find P(event A or event B)

Solution

Possible outcomes	1	2	3	4	5	6	7	8	9	10
Event A		2		4		6		8		10
Event B	1	2	3	4	5	6				
Event A or Event B	**1**	**2**	**3**	**4**	**5**	**6**		**8**		**10**

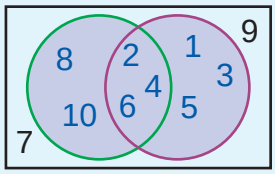
There are: 10 possible outcomes altogether,
5 favorable outcomes for event A,
6 favorable outcomes for event B.

But you **can't** add 5 to 6 to find the number of favorable outcomes for **A or B**. This is because 3 outcomes match **both** events, and you don't count these twice.

In fact there are **8 favorable outcomes** — **2** match event A only, **3** match event B only, and **3** match both events.

So P(even number or less than 7) = $\frac{8}{10}$ = **80%**

Check it out:
In this example, we are looking for the outcomes that are in either of the two circles in the Venn Diagram.



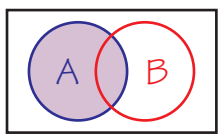
Guided Practice

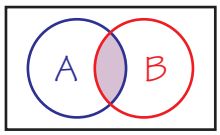
5. Using the pairs of events from Exercises 1–4 for the spinner shown above, find P(event A or event B).

Venn Diagrams Can Show You How to Find P(A or B)

You might find it helpful to use Venn diagrams to picture what happens when you combine outcomes.

To find P(A), you count all the outcomes in circle A.

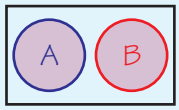




To find P(A and B), you count all the outcomes that are in both circles at the same time.

Check it out:

If the events have no outcomes in common, there is no overlap.



So P(A and B) = 0
and P(A or B) = P(A) + P(B)

Check it out:

You can use the rule above to show that the formula you met in the last Lesson is true: "P(A) + P(not A) = 1."

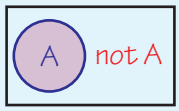
(i) P(A OR not A) = 1
(since an event either happens or it doesn't).

(ii) P(A AND not A) = 0
(since an event can't both happen and not happen).

Substitute these numbers into
P(A or B) = P(A) + P(B)
– P(A and B)
using "not A" as event B.

You get:
1 = P(A) + P(not A) – 0.
Or 1 = P(A) + P(not A).

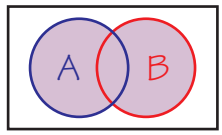
The Venn diagram would look like this:



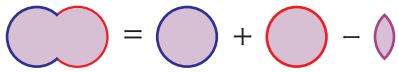
Now try these:

Lesson 6.2.6 additional questions — p460

To find P(A or B) you count any outcomes that are in either circle, but you only count the outcomes in the overlap once.



So P(A or B) = P(A) + P(B) – P(A and B)



because P(A) + P(B) counts the overlap twice — once in A and once in B.

Guided Practice

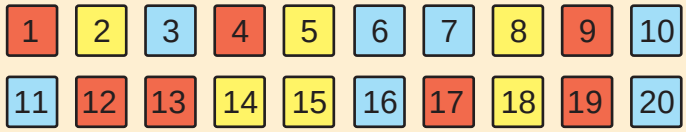
Aisha takes a standard pack of 52 playing cards.
She picks one card at random. Calculate the following probabilities.

6. P(red)
7. P(5)
8. P(red and 5)
9. P(red or 5)
10. P(black)
11. P(face card)
12. P(black and a face card)
13. P(black or a face card)
14. P(hearts)
15. P(clubs)
16. P(hearts and clubs)
17. P(hearts or clubs)

18. For two events, A and B, P(A) = 25%, P(B) = 30%, and P(A or B) = 50%. Are there any outcomes that match both events? Explain your answer.

Independent Practice

Bernardo picks one of the cards shown here at random.
Calculate the following probabilities. Write your answers as decimals.



1. P(odd)
2. P(even)
3. P(red)
4. P(blue)
5. P(yellow)
6. P(odd and blue)
7. P(odd or blue)
8. P(red and yellow)
9. P(red or yellow)
10. P(yellow and even)
11. P(yellow or even)
12. P(red and a multiple of 4)
13. P(blue and prime)
14. P(yellow or less than 10)

15. Sally spins the spinner shown in Examples 1 and 2.
She calculates P(even or less than 5) = 90%.
What mistake has Sally made? What is P(even or less than 5)?

Round Up

The tricky thing to remember from this Lesson is what "or" means in probability.
"A or B" doesn't mean "either A or B" — it means "either A or B or both."
If you forget about that "or both" bit, you'll wind up getting questions wrong.

Section 6.3 introduction — an exploration into:

Pick a Card

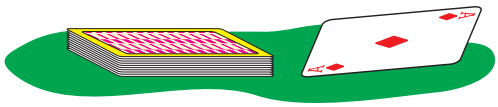
You're going to conduct two probability experiments with cards. The experiments are very similar, but there's one big difference between them. As you do the Exploration, think about that difference, and the effects it has on your two sets of results.

Before you do your two experiments, you need to create a **table** to record your **results**. Copy the table on the right.

Now you need some **playing cards**.
You will need **one complete suit** of **13** playing cards.
Shuffle them, and put them in a pile (with the cards **face down**).

You need the Ace-King of any suit.

	Experiment 1	Experiment 2
Pick #	Card	Card
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		



Experiment 1

1) Choose **one card** from the pile. You can choose any card you like.
2) **Record** the result in your table.
3) Now put that card **to one side**. **Do not put it back in the pile.**
4) Repeat steps 1–3 until you have picked **13 times**. There will be **one less card** to choose from each time. For your final pick, there will only be **one** card left.

Experiment 2

Use the same set of 13 cards. **Shuffle** them, and place them face down in a pile.

1) Choose **one card** from the pile. You can choose any card you like.
2) **Record** the result in your table.
3) **Put the card back in the pile and reshuffle the cards.**
4) Repeat steps 1–3 until you have picked **13 times**. There will **always** be **13 cards** to choose from.

Exercises

1. Did you pick the same card twice during either experiment? Explain why/why not.
2. Now think about the probabilities of choosing cards in Experiment 1.
 a. What was the probability you would choose the Ace on your 1st pick?
 b. If you had chosen the Ace on your 1st pick, what was the probability you would choose the Ace on your 2nd pick? And on the 3rd pick? On the 8th pick? On the 13th pick?
3. Now think about the probabilities of choosing cards in Experiment 2.
 a. What was the probability you would choose the Ace on your 1st pick?
 b. If you had chosen the Ace on your 1st pick, what was the probability you would choose the Ace on your 2nd pick? And on the 3rd pick? On the 8th pick? On the 13th pick?

Round Up

This Exploration was all about how events can be dependent or independent — it's a really important idea in probability. You're going to learn much more about it in this Section.

Lesson 6.3.1

Section 6.3

Independent and Dependent Events

California Standard:
Statistics, Data Analysis and Probability 3.5
Understand the difference between independent and dependent events.

What it means for you:
You'll learn the difference in math between events that don't affect what happens later, and those that do.

Key words:
- independent events
- dependent events
- probability
- event

In Section 6.2, you came across the idea that you can figure out the probability that an event will happen. This Lesson is all about how different events can affect each other — how one event happening can affect the probability of another event happening.

One Event Happening Can Affect Later Probabilities

A really important idea in probability is whether events are **independent** or **dependent**. Imagine Luis and Gabrielle both belong to a gym...

(i) Each morning when he wakes up, Luis decides **completely at random** whether he'll go to the gym that day or not.
Every day, the probability that Luis will go to the gym is **0.7**.

(ii) Gabrielle has a different system. If Gabrielle has been to the gym the previous day, she is **less likely** to want to go the next day too.
If she **went** to the gym yesterday, the probability she will go to the gym today is **0.1**. But if she **didn't go** to the gym yesterday, then the probability she will go today is **0.8**.

Other events **do not affect the probability** that Luis will go to the gym, so this event is **independent** of others.

However, the probability of the event "Gabrielle will go to the gym today" is **dependent** on the event "Gabrielle went to the gym yesterday."

Check it out:
In the gym example, Luis decided at random what to do. Events that happen at random are always independent.

Independent Events

Two events are independent if one event happening doesn't affect the probability of the other event happening.

In other words, independent events have no influence on each other.

Example 1

Which of the following pairs of events are independent, and which are dependent?

(i) "Throwing a head on a coin," and "throwing a 6 on a die."
(ii) "Throwing a head on a coin on the first spin," and "throwing a head on a coin on the second spin."

Solution

(i) The result of the coin toss doesn't affect the result of throwing the die, so these events are **independent**.
(ii) The result of the first coin toss doesn't affect the result of the second toss, so these events are also **independent**.

Guided Practice

Check it out:
For events to be independent, the result of the first should have no effect at all on the result of the second.

1. Mr. Garcia has an apple, a banana, and a peach. He randomly picks one piece of fruit and eats it. Then he picks another and eats it.
Are the events "eats the apple first" and "eats the apple second" independent? Explain your answer.

2. Every day after school Dominic goes running, cycling, or swimming. He chooses randomly which activity to do.
Are the events "Dominic runs on Monday" and "Dominic runs on Tuesday" independent? Explain your answer.

3. The basketball squad need to choose a captain and a co-captain. They pick a captain from the team. From the remaining players, they then select a co-captain. Are the events "Letisha becomes captain" and "Letisha becomes co-captain" independent?
Explain your answer.

Coins and Spinners Give Independent Events

Events based on tossing coins and spinning spinners are always **independent**. The result of one coin toss, or one spin on a spinner, doesn't affect what happens on later tosses or spins.

Check it out:
The example talks about a fair coin. This means it's just as likely to land on heads as tails. It's not biased in any way.

Example 2

You toss a fair coin three times. On your first toss, it lands on heads.
What is the probability of the coin landing on heads with:
(i) your second toss? (ii) your third toss?

Solution

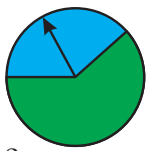
Coin tosses are always **independent**. This means that the result of one coin toss doesn't affect the probability of any of the later results.

Whenever you toss a fair coin, heads and tails are both **equally likely**.
So for a fair coin, P(heads) = 0.5 and P(tails) = 0.5.
(i) This means that for your second toss: **P(heads) = 0.5**.
(ii) This also means that for your third toss: **P(heads) = 0.5**.

Check it out:
Here, the events "spinner lands on blue" and "spinner lands on green" are <u>not equally likely</u>.
However, they <u>are independent</u> — the result of one spin does not affect the probability of getting blue or green on later spins.

Example 3

You spin the spinner shown on the right two times.
The probability of the spinner landing on blue is 0.35.
The first two spins land on blue.
What is the probability that the third spin also lands on blue?



Solution

The spins of a spinner are independent. So it doesn't matter what the first two results were, the probability of it landing on blue with the third spin must be **0.35**.

Guided Practice

Say whether Events A and B in Exercises 4–7 are independent.

4. Kalisa tosses a coin twice.
Event A: She tosses a head with her first throw.
Event B: She tosses a tail with her second throw.

5. Jordan rolls a number cube twice.
Event A: He throws a 2 on the first throw.
Event B: He throws a number less than 3 on the second throw.

6. Madison spins the spinner shown two times and gets purple on both spins. He then spins the spinner two further times.
Event A: He gets purple with his third spin.
Event B: He gets purple with his fourth spin.



7. Ethan has two bags, each with 5 numbered chips in it. He picks one chip from each bag.
Event A: He picks the chip numbered 2 from the first bag.
Event B: He picks the chip numbered 2 from the second bag.

Independent Practice

Explain your answer to each of Exercises 1–3.

1. Maria likes both fish and carrots, but she never eats them in the same meal. Are the events "eats fish for her evening meal" and "eats carrots with her evening meal" independent?

2. You spin a spinner numbered 1–10, then toss a coin, and then roll a number cube. Are the events "spin a 7," "throw a tail," and "throw a 4" independent?

3. There are 23 instruments. The 23 class members take it in turn to pick an instrument each to play. Are the events "the first student picks a drum" and "the second student picks a flute" independent?

Say whether Events A and B in Exercises 4–5 are independent.

4. You toss a coin 10 times.
Event A: the 9th toss is a head.
Event B: the 10th toss is a head.

5. Ted has five different-colored pens. Each day, he picks one to take to school, but he never takes the same pen more than once in a week.
Event A: He takes the red pen on Monday.
Event B: He takes the blue pen on Tuesday.

Now try these:
Lesson 6.3.1 additional questions — p460

Round Up

If one event has no effect at all on the probability of another event, then they are independent of each other. It doesn't mean that the probabilities have to be equal — remember the spinner in Example 3 — they just have to be unaffected by what's gone before. There's more about this next Lesson.

Lesson 6.3.2

Events and Probabilities

In the last Lesson, you saw that some events affect what happens later, while others don't. This Lesson talks about dependent and independent events in a bit more detail.

California Standard:
Statistics, Data Analysis and Probability 3.5
Understand the difference between independent and dependent events.

What it means for you:
You'll learn how to recognize when events are affecting each other by looking at probabilities.

Key words:
- dependent event
- independent event
- outcome
- probability
- replacement

Don't forget:
You saw in Section 6.2 that the formula for calculating the probability of an event happening is:

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Possible outcomes}}$$

If you need a reminder, you can see more about this in Lesson 6.2.2.

Probabilities of Independent Events Don't Change

To decide if events are independent, you can look at **probabilities**. If one event's probability is affected by another event, they're **dependent**.

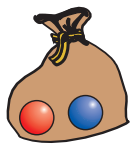
Example 1

There are 2 red marbles and 1 blue marble in a bag. You pick a marble, **put it to one side**, and then pick another. Show that the events "picking a red marble with the first pick" and "picking a red marble with the second pick" are dependent.



Solution

Suppose you pick a **red** marble with your first pick. Then there is one red marble and one blue marble left in the bag. So **P(pick red with second pick) = $\frac{1}{2}$**.



Now suppose you **don't** pick a red marble with your first pick. This means there are two red marbles left in the bag. So **P(pick red with second pick) = 1**.



Whether the first event happens or not **affects the probability** of the second event. So the two events are **dependent**.

This next example is identical, except the first marble is **put back**.

Example 2

There are 2 red marbles and 1 blue marble in a bag. You pick a marble, **put it back**, and then pick another. Show that the events "picking a red marble with the first pick" and "picking a red marble with the second pick" are independent.



Solution

Suppose you pick a **red** marble with your first pick, then put it back. There will still be two red marbles and one blue marble in the bag. So **P(pick red with second pick) = $\frac{2}{3}$**.



Now suppose you pick the **blue** marble with your first pick, then put it back. There will still be two red marbles and one blue marble in the bag. So **P(pick red with second pick) = $\frac{2}{3}$**.



Whether the first event happens or not **does not affect the probability** of the second event. So the two events are **independent**.

Guided Practice

In Exercises 1–3, calculate the probability of Event B twice:
(i) when Event A has happened, and (ii) when Event A did not happen.
Use your answers to decide whether Events A and B are dependent.

1. Calvin has a bag with six counters in it, numbered 1–6. He pulls out a counter and puts it to one side. Then he pulls out another counter.
Event A: He pulls out the counter numbered 1 on his first pick.
Event B: He pulls out the counter numbered 1 on his second pick.

2. Madison spins the spinner shown on the right 2 times.
Event A: She spins purple with her first spin.
Event B: She spins green with her second spin.



3. Jordan rolls a number cube twice.
Event A: He gets a 4 with his first roll.
Event B: He gets an odd number with his second roll.

Selection with Replacement Gives Independent Events

A lot of the probability situations you'll meet in this grade are connected with selecting various objects at random.

Once you've made your first pick, there are two things you can do before you make your second pick:
(i) you can **put the first object back**, or
(ii) you can make your second pick **without** putting the first pick back.

Putting the first pick back means your two selections are **independent**.

Don't forget:
Examples 1 and 2 also show how replacing (or not replacing) the first item affects probabilities for the second pick.

Example 3

Kalisa picks a card from a standard deck, and then puts it back. She then picks another card.

Show that the events "picking a red card with her first pick," and "picking a black card with her second pick," are independent.

Solution

Since Kalisa has replaced her first card, **it doesn't matter** what she got with her first pick. Either way, the deck will contain **exactly the same** cards for her second pick.

Since there are 52 cards in total, and 26 of them are black, the probability that she gets a black card with her second pick is:



$$P(\text{black}) = \frac{26}{52} = \frac{1}{2}$$

If the first card is **not** put back, then the deck has changed for the second pick. This means that the probabilities for the second pick will change.

So not replacing your first pick will always lead to **dependent** events.

Example 4

Kalisa picks a card from a standard deck, and then puts it to one side. She then picks another card.

Show that the events "picking a red card with her first pick," and "picking a black card with her second pick," are dependent.

Solution

Suppose Kalisa picks a **red** card with her first pick. For her second pick, there will now be 51 cards — 26 black and 25 red. So P(black) = $\frac{26}{51}$.



But if she picks a **black** card with her first pick, then for her second pick, there will still be 51 cards — but this time, 25 black and 26 red. So P(black) = $\frac{25}{51}$.



As these probabilities are different, the two events must be **dependent**.

Check it out:

If the first selection is not replaced, this affects what you can choose from for your second pick.
This means probabilities for your second pick are affected by what happens with your first pick, which makes events dependent.

Guided Practice

4. Kadema has 26 cards in a bag, each marked with a different letter of the alphabet. He picks one letter, puts it to one side, then draws out a second letter. Show that the events "picks a vowel with first pick" and "picks a consonant with second pick" are dependent.

In Exercises 5–6, say whether the selections are independent.
5. Melisenda picks a four-digit combination for her school locker. She chooses each digit at random.
6. Corran also picks a four-digit combination for his locker. He decides not to use each digit more than once.

Don't forget:

Vowels are A, E, I, O, and U. All other letters are consonants.

Independent Practice

1. Necie has a bag with six counters in it, numbered from 1 to 6. He pulls out a counter, keeps it and then pulls out another. By calculating probabilities, show that the events "picks odd number with first pick" and "picks even number with second pick" are dependent.

In Exercises 2–3, say whether the selections are dependent.
2. Each month, the class picks a different student to be a hall monitor.
3. Jessica is holding three long straws and one short straw. The ends are hidden so that nobody can see which is the shortest. One at a time, Jessica's four friends draw a straw until Jessica has no straws left.

4. When you pick items at random from a selection, why do they have to be put back in for the events to be independent?

Now try these:

Lesson 6.3.2 additional questions — p461

Round Up

To decide whether events are dependent or independent, you have to look at probabilities. But... there's sometimes a shortcut — selecting without replacement always leads to dependent events.

Lesson 6.3.3

Calculating Probabilities of Independent Events

California Standard:

Statistics, Data Analysis, and Probability 3.4

Understand that the probability of either of two disjoint events occurring is the sum of the two individual probabilities and that the **probability of one event following another, in independent trials, is the product of the two probabilities.**

What it means for you:

You'll see that there's a formula you can use to figure out the probability that two events will both happen.

Key words:

- independent
- event
- probability
- outcome

Don't forget:

Example 1 uses the same method as Section 6.2.

You've already seen in Section 6.2 how to figure out the probability that an event will happen. And you saw in the last Lesson how you can use those probabilities to show that two events are independent. In this Lesson, you'll see a formula to work out the probability that two independent events happen in a row.

You Can Show Outcomes for Two Actions in a Table

Think about **rolling two dice**. What's the **probability** of **rolling two numbers higher than 3**?



Example 1

You have a red die and a blue die.
Draw a table showing all the possible outcomes of rolling these two dice.
Shade the outcomes that show rolling two numbers higher than 3.
What is the probability of rolling higher than 3 on both dice?

Solution

First **draw your table**.

Blue die \ Red die	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The purple boxes show outcomes greater than 3 on both rolls.

Bottom 3 rows show a number greater than 3 on blue die.

3 columns on the right show a number greater than 3 on red die.

The **probability** of rolling greater than 3 on both dice is:

$$P(\text{red die} > 3 \text{ and blue die} > 3) = \frac{\#\text{ favorable outcomes}}{\#\text{ possible outcomes}} = \frac{9}{36} = \frac{1}{4}$$

Guided Practice

1. How many ways can you score over 3 using a single die?

2. How many ways can you get two scores over 3 using two dice?

3. What is the probability of scoring over 3 using a single die?

4. What is the probability of getting two scores over 3 using two dice?

5. When you roll two dice, are the events "roll over 3 on die 1" and "roll over 3 on die 2" independent? Justify your answer.

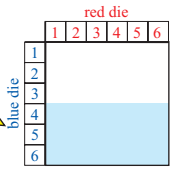
6. What do you notice about your answers to Exercises 1 and 2? And what do you notice about your answers to Exercises 3 and 4?

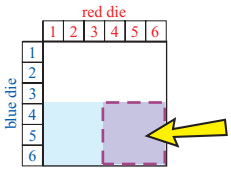
Multiply Probabilities If Events Are Independent

There's actually a quicker way to work out the probability in Example 1. It means you don't have to draw a table.

To begin with, just **imagine** the table showing all the possible results for throwing the two dice.

The probability of getting more than 3 on the blue die is $\frac{1}{2}$.





The probability that any of these rolls will be followed by more than 3 on the red die is also $\frac{1}{2}$.

To find the fraction of the possible outcomes that show over 3 on both dice, you've found $\frac{1}{2}$ of $\frac{1}{2}$. This means that the probability of getting over 3 on both dice must be $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ — you've **multiplied probabilities**.

In fact, for any two **independent** events...

$$P(A \text{ and } B) = P(A) \times P(B)$$

Don't forget:
P(get over 3 on a die) = $\frac{1}{2}$.

Don't forget:
These events are independent. Look back to Example 1 of Lesson 6.3.1 if you don't remember what this means.

Example 2

What is the probability of:
(i) Throwing a head on a coin, and then throwing a 6 on a die?
(ii) Throwing a head on a coin on the first toss, and then throwing another head on the coin on the second toss?

Solution

(i) P(head on a coin) = $\frac{1}{2}$, and P(6 on a die) = $\frac{1}{6}$.

So P(head and a 6) = $\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$

(ii) P(head on a coin) = $\frac{1}{2}$, so P(a head on two tosses) = $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Sometimes, you have to use the formula the "wrong way around."

Example 3

The probability of a spinner landing on red is 0.25.
The probability of spinning red and then green is 0.0625.
What is the probability of spinning green?

Solution

Use a variable (x, say) for the probability of the spinner landing on green.
This means P(green) = x, and P(red) = 0.25.

These events are independent, so P(red, then green) = $\mathbf{0.25}x$.
But you also know that P(red, then green) = **0.0625**.

Put these together and you find that $\mathbf{0.25}x = \mathbf{0.0625}$.
So $x = 0.0625 \div 0.25 = 0.25$ **Divide both sides by 0.25 to find x.**
So the **probability of spinning green is 0.25**.

Guided Practice

Exercises 7–14 are about the bag of marbles on the right.



You draw a marble from the bag shown, note its color, and replace it. Then you draw another one.

7. Will the result of the first pick affect the probability of picking a particular color with the second pick? Explain your answer.

8. Are the two draws independent or dependent?

In Exercises 9–14, find the probabilities of the given events.

9. P(pick green)
10. P(pick blue)
11. P(pick red)
12. P(pick green, then red)
13. P(red, then green)
14. P(blue, then red, then green)

15. Pepe picks a hat and scarf to wear at random from his drawer. The probability he picks a red hat is 0.4. The probability he picks a red hat and a blue scarf is 0.1. Find the probability he picks a blue scarf.

Check it out:
In Exercise 15, you know P(A and B), and you have to find P(B).

Independent Practice

In Exercises 1–6, find the probability of rolling the numbers shown on two number cubes.

1. P(6 and 6)
2. P(1 then 6)
3. P(even number, then odd number)
4. P(odd number, then less than 3)
5. P(less than 6, then less than 2)
6. P(more than 2, then less than 4)

7. Noah picks a red card from a standard deck. He keeps it and picks another, which is also red. He says the probability of this is $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$. Is he correct? Explain your answer.

8. Ceri draws a marble from one bag, and a shape from another bag. The probability of picking a green marble and then a square is $\frac{5}{24}$. If the probability of picking a green marble is $\frac{5}{6}$, what is the probability of picking a square?

9. Dominique has a bag of nine marbles, of which two are red. She draws a blue marble, puts it back, and then draws a red marble. The probability of this combination of events was $\frac{8}{81}$. Calculate the number of blue marbles in the bag.

Now try these:
Lesson 6.3.3 additional questions — p461

Round Up

If two events are independent, then you can find the probability that they both happen by multiplying probabilities together. This can save you lots of time, because it means you don't have to draw tables or tree diagrams.

Section 6.4 introduction — an exploration into:
Shooting Baskets

You can use the past to predict the future — that much is almost common sense. But doing it accurately is where you need math. That doesn't mean you can say for sure exactly what's going to happen in the future, but it does mean you don't have to make a wild guess.

For this Exploration you need to work in **teams**.
Your team is going to try to score "**baskets**" by throwing balls of **paper** into the **trashcan** from: (i) **2 meters**, and (ii) **5 meters**.



First, the team will take a **total of 20 shots** from each distance.

So if there are **4 people** in your team, each person takes **5 shots**.

Before starting to throw, everyone has to make a **prediction** of how many shots the team will make.

Record your predictions in a **table** like this one.

Distance from trash can	Prediction for 20 shots	Actual result for 20 shots
2 meters		
5 meters		

The team should now take 20 shots from each distance. Record the results in your table.

Exercises

1. Calculate:
- **a.** the fraction of baskets that you predicted the team would make,
- **b.** the fraction of baskets that the team actually made.

Divide the **number** of baskets predicted/made by **20**.

2. How close was your prediction?

Subtract the percent **made** from the percent **predicted**.

The team is now going to take **30 extra shots** from: (i) **2 meters**, and (ii) **5 meters**.

Before throwing, each person must again **predict** how many baskets the team will make.
Use your data from the first 20 shots at each distance to help.

Extend your table to record your predictions and the results.

Prediction for 30 shots	Actual result for 30 shots

The team should now take 30 shots from each distance. Record the results in your table.

Exercises

3. Explain how you used the data from the first 20 shots to make predictions for 30 shots.

4. Subtract the fraction made from the fraction predicted to see how good the predictions were.

5. How many shots would you predict the team would make if it took 100 shots?
You can use the data from your tables. Explain how you made your estimate this time.

Round Up

Did the data from the first 20 shots help you to make a better prediction of what would happen on the next 30? Even if you didn't predict exactly what would happen on the 30 shots, making the second prediction was hopefully easier than the first (which was probably just a guess).

Lesson 6.4.1

Relative Frequency

California Standard:

Statistics, Data Analysis and Probability 3.2

Use data to estimate the probability of future events (e.g., batting averages or number of accidents per mile driven).

What it means for you:

You'll see how to use the results of experiments that you've already done to predict the results of future experiments.

Key words:

- relative frequency
- experimental probability
- outcome
- trial

Check it out:

The word "trial" means an action. So tossing a coin once is one trial, tossing a coin twice is two trials, and so on. The word "experiment" can also be used.

Check it out:

Like a probability, relative frequency can be a fraction, a decimal, or a percentage. So you could write the answer to Example 1 as $\frac{53}{100}$, 0.53, or 53%.

Check it out:

If you work out the relative frequencies of all the possible results, then they should add up to 1 — just like probabilities.

In Section 6.2, you saw how to work out the probability that an event will happen by looking at how likely different possible outcomes are. Relative frequency is another kind of probability measure — but it's based on looking at outcomes from the past.

Relative Frequency Shows Past Results

If you **repeat** an action such as tossing a coin or spinning a spinner a number of times, you can look at **how often** one particular **outcome** occurs. This is called the **relative frequency** of the outcome.

You calculate relative frequency by **dividing** the **number of times** you saw one **outcome** by the **number of times** that you **did the experiment**.

Relative Frequency:

$$\text{Relative Frequency} = \frac{\text{number of times outcome occurred}}{\text{number of trials}}$$

Example 1

You toss a coin 100 times. It lands heads 47 times, and tails 53 times. Find the relative frequency of the coin landing on tails.

Solution

$$\text{Relative Frequency} = \frac{\text{number of times outcome occurred}}{\text{number of trials}}$$

$$= \frac{53}{100}$$

Example 2

You roll a number cube 500 times. You get the following results.

Score	1	2	3	4	5	6
Frequency	89	82	85	77	88	79

Calculate the relative frequency of each score to 3 decimal places.

Solution

To calculate a score's relative frequency, divide the frequency by the total number of rolls.

For example, the relative frequency of 1 is $\frac{89}{500} = 0.178$

Doing the same for the other scores gives the following:

Score	1	2	3	4	5	6
Relative Frequency	0.178	0.164	0.170	0.154	0.176	0.158

Guided Practice

Find the relative frequencies described in Exercises 1–3.

1. You toss a coin 56 times. Heads comes up 23 times.
Find the relative frequencies of heads and tails.

2. A 3-section spinner was spun 99 times.
Blue came up 37 times, red 58 times, and green the remainder.
Find the relative frequency of each color.

3. For a probability experiment, Tim picked a card from a standard deck, recorded the suit, and then replaced the card. He then repeated this until he had picked a total of 200 times. His results are shown in the table.
Find the relative frequency of each suit.

Hearts	58
Diamonds	43
Clubs	51
Spades	48

Relative Frequency Can Be Used as a Probability

Relative frequency can be used to estimate **probability**.
When you do this, you're using the **past** to predict the **future**.
This kind of probability is called an **experimental probability**.

Check it out:

A theoretical probability is one that you work out by comparing how likely it is that each possible outcome of a trial will happen. You can say that, since a coin has two sides, the theoretical probability of a tail is 0.5.

An experimental probability is one that you calculate by working out how often an outcome has occurred in past experiments — its relative frequency. You can say that if you flip a coin 100 times and get 52 tails, the experimental probability of a tail is 0.52.

Example 3

The spinner on the right was spun 1000 times.
The results are shown in the table.



Use these results to estimate the probability of the next spin landing on: (i) red, (ii) green.

Red	514
Green	221
Blue	265

Solution

Relative frequency can be used as a probability.

(i) Use the relative frequency of red.
So the experimental probability of red is $514 \div 1000 = \mathbf{0.514}$.

(ii) Use the relative frequency of green.
So the experimental probability of green is $221 \div 1000 = \mathbf{0.221}$.

Relative frequency can be used as a probability in all sorts of situations.

Example 4

In a medical study of 1000 people suffering from a particular condition, 630 patients saw an improvement after taking a new drug.
Based on these results, estimate the probability that another patient with the same condition will see an improvement after taking the new drug.

Solution

Use relative frequency as a measure of probability.

$$\text{Relative frequency} = \frac{\text{Number of people improved}}{\text{Number of people in study}} = \frac{630}{1000} = \mathbf{0.63}$$

P(another patient improving) = 0.63

Guided Practice

4. A school is looking at the data concerning where its pupils live. The school has 1121 pupils altogether, and 267 of them come from Town A. Based on this, what is the probability that the next student to enroll at the school will live in Town A?

5. A school wants to start a new sports club. It takes a poll of the students to see which sport is most popular. The numbers of votes so far for each sport are shown in the table on the right. One more student is still to vote. Use the results to find the probability this student will vote for each sport.

Sport	Votes
Tae kwon do	219
Lacrosse	113
Surfing	375

6. The Green Club asks members how many aluminium cans their families had bought and how many of these they had recycled, during the past month. They found that 247 cans had been bought, of which 229 had been recycled, the rest thrown away. Estimate the probability that the next can bought by a member will be thrown away.

Independent Practice

1. A spinner is spun 100 times, and lands on red 27 times. Calculate the relative frequency of landing on red.

2. You plan to toss a coin 25 times. What should you do to find the relative frequency of heads?

3. Peppa spins a triangular spinner 20 times. It lands on red 7 times and lands 5 times on blue. If Peppa spins the spinner again, estimate the probability that it will land on its third side – green.

4. Pepe does a survey of the color of cars that pass his house in one hour. His data is shown in the table on the right. Calculate the relative frequency of each car color.

Car Color	Frequency
Red	23
Blue	7
Silver	16

5. A factory randomly tests its goods for faults. The whole day's output is tested if the probability of getting a faulty item is greater than 0.02. On day A, 21 items out of 948 randomly tested were faulty. On day B, 14 out of 999 tested were faulty. Does the whole day's output for either of these days need to be tested again? If yes, then which day(s)?

6. A 3-section spinner was spun 25 times, giving the results shown in the table on the left. Fill in the frequency of each color.

Color	Frequency	Relative Frequency
Green		12%
Orange		60%
Purple		28%

Don't forget:

The frequency of an outcome is just how many times it happens. So if you see 5 red cars in an hour, you can say that the frequency of red cars in your survey is 5.

Now try these:

Lesson 6.4.1 additional questions — p461

Round Up

You can use relative frequency as a measure of how likely something is to happen. And you can use it for events where you couldn't work out a probability in other ways. It's a way of using your experience of the past to predict an outcome in the future. You'll see more about all this in the next Lesson.

Lesson 6.4.2

Making Predictions

Experimental probability is all about using the relative frequency of an outcome to work out the probability that the same outcome will happen again. Like all probability, it's about making predictions.

California Standard:

Statistics, Data Analysis and Probability 3.2

Use data to estimate the probability of future events (e.g., batting averages or number of accidents per mile driven**).**

What it means for you:

You'll see how to use the results of experiments that you've already done to predict the results of future experiments.

Key words:

- experimental probability
- theoretical probability
- relative frequency
- estimate

Check it out:

If a die is "biased," it means it doesn't behave like a die should — for example, maybe it comes down on "1" much more often than you'd expect. In the same way, a coin that landed on heads 80 times out of 100 is probably biased — you'd expect about 50 heads, and 80 is a lot more than this. A coin/die that isn't biased is usually described as either "fair" or "unbiased."

Don't forget:

This is the same as with theoretical probability — multiply the number of trials by an event's probability to predict the number of times that event will happen. Experimental probability is useful in areas like business and sports — where theoretical probability is tricky to calculate and lots of old data is available.

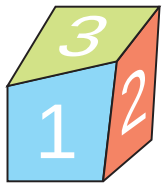
Create Your Own Data to Estimate Probability

You can use **experimental probability** to **predict** the **outcome** of an experiment when you can't use **theoretical probability**.

For example, using experimental probabilities might be the only way you can make predictions about a **biased** die.

Example 1

Pepe has an odd-shaped die, which he thinks will be biased. Describe how he could find the probability of scoring a 3 using this die.



Solution

Pepe would need to undertake some trials. He should roll the die, say, 100 times, and count the number of times he scored 3.
He can then work out the **relative frequency**.

For example, if he scores 3 on 40 occasions, he would say that the **probability** of scoring 3 in the future would be: $\frac{40}{100} = \frac{2}{5}$

Guided Practice

1. Sandi rolled a biased die 150 times and found that it landed on the number 4 a total of 40 times. What is the experimental probability that the coin will land on 4 with the next throw?

2. Juan stuck some modeling clay to one side of a coin and tossed it 20 times. He found that it landed on heads 15 times. Find the experimental probability that the coin will land on heads next time.

Use Experimental Probabilities with Real-Life Events

You can use relative frequency to estimate the number of times something will happen in the future.

Example 2

At a factory, the relative frequency of tires failing a safety test is 0.01. Estimate how many tires will fail the test in a batch of 4000.

Solution

The experimental probability that a tire will fail is 0.01.
Multiply the number of tires by this probability to estimate the number that will fail: $4000 \times 0.01 =$ **40 tires**

Don't forget:
To work out a batting average, divide the number of hits by the number of times at bat.

Baseball **batting averages** are examples of **relative frequency**, or **experimental probability**.

Example 3

Barry has had 15 hits in 50 times at bat.
What is his batting average?

If he has the same batting average for his next 30 at bats, how many hits will he get?

Solution

Batting average = Hits ÷ Number of at bats
= 15 ÷ 50 = **.300**

To find the **number** of hits in Barry's **next 30** at bats, **multiply 30** by his **batting average**: 30 × .300 = **9 hits**

Guided Practice

3. On the basketball court, Kobe made 17 shots out of 20. Use relative frequency to estimate how many baskets he will make in 300 shots.

4. In a survey of 50 students, 23 said that they rode the bus to school. If these 50 students are representative of the whole school, how many students out of 1350 would you estimate will ride the bus to school?

5. In the past, School A's football team has beaten School B's team in 40% of their games. Based on these results, how many of next season's 5 games would you estimate School A will win?

The More Data You Have, the Better

When you work out **experimental probability**, the **more trials** that you do, the **more accurate** your estimate is likely to be. A probability based on 1000 trials is likely to be more accurate than one based on 10.

Example 4

Alan rolls a fair number-cube 10 times, and records the scores.
He then rolls it a further 90 times and records these scores too.
Calculate the experimental probability of each score based on:
(i) 10 rolls, (ii) 100 rolls.

Score	1	2	3	4	5	6
In 10 rolls	2	0	1	3	3	1
In 100 rolls	19	17	15	14	17	18

Which set of probabilities is likely to give more accurate predictions?

Solution

You can use a table.

Score	1	2	3	4	5	6
Relative Frequency after 10 rolls	2÷10 = **0.2**	0÷10 = **0**	1÷10 = **0.1**	3÷10= **0.3**	3÷10 = **0.3**	1÷10 = **0.1**
Relative Frequency after 100 rolls	19÷100 = **0.19**	17÷100 = **0.17**	15÷100 = **0.15**	14÷100 = **0.14**	17÷100 = **0.17**	18÷100 = **0.18**

The probabilities based on **100 rolls** are likely to give better predictions.

Don't forget:
To work out the experimental probability of rolling a 6, divide the number of favorable outcomes by the total number of possible outcomes. See Lesson 6.2.2 for more information.

Example 5

Score	1	2	3	4	5	6
In 1000 rolls	171	164	166	170	161	168

Alan then rolls his number cube a further 900 times.
He now has a total of 1000 results.
How do his results for 10 rolls, 100 rolls, and 1000 rolls compare with the theoretical probability for each score?

Solution

Find the experimental probabilities for 1000 rolls in exactly the same way as for Example 4.

Score	1	2	3	4	5	6
Relative Frequency after 1000 rolls	0.171	0.164	0.166	0.17	0.161	0.168

The **theoretical probability** of each score is $1 \div 6 = 0.1666... = \mathbf{0.16\overline{6}}$.

The greater the number of rolls, the closer most of the experimental probabilities are to the theoretical probability.

Check it out:
As you get more and more data, your experimental probabilities should get closer and closer to the theoretical probabilities.

Guided Practice

Sarah has a fair five-color spinner, as shown.



6. What is the theoretical probability of it landing on each color?

7. She spins it 12 times and gets the results in the table.

Colour	Red	Blue	Yellow	Purple	Green
Frequency	1	1	3	2	5

Find the experimental probability for each color, using these results.

8. How would you expect her experimental probabilities to change if she spun the spinner 1000 times?

Don't forget:
You can use experimental probability to predict the outcome of any event. But it's most useful when you're looking at real-life events whose theoretical probabilities are impossible to calculate.

Independent Practice

In Exercises 1–4, use the relative frequency given to estimate the number of times that event A will occur in 20 trials.

1. $P(A) = 0.2$ **2.** $P(A) = 30\%$

3. $P(A) = \frac{1}{5}$ **4.** $P(A) = 0$

Manny has a batting average of .350. Use this to find:

5. the expected number of hits if he batted 100 times.

6. the expected number of hits if he batted 40 times.

7. the expected number of hits if he batted 220 times.

8. Mrs. Hill cycles to work. She notes what color the traffic lights are when she reaches them. She finds they are red on two-fifths of the times she reaches them. Based on these figures, how many times in 20 trips are the lights likely to be red when she arrives?

Now try these:
Lesson 6.4.2 additional questions — p462

Round Up

You can use experimental probability to predict the outcome of an event based upon data you've collected previously. This is all about using information from the past to predict the future.

Chapter 6 Investigation
A Game of Chance

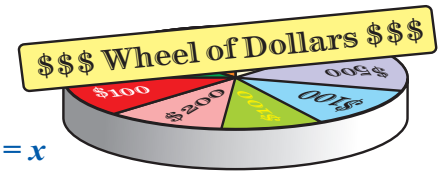
The producers of a new TV game show, "Wheel of Dollars," need a wheel for contestants to spin. The wheel will be divided up into equally-sized sections, and each section marked with a prize.

Part 1:

The producers need the wheel to be designed so that the probabilities of winning the amounts shown below are:

- **P(win $100) = 0.35**
- **P(win $200) = 0.3**
- **P(win $300) = 0.2**
- **P(win $500) = 0.1**
- **P(win $1000) = *x***

(There are no other possible outcomes.)



Design a wheel for use on the game show.

Remember... each section on the wheel must be the same size.

Things to think about:

- What is the probability of winning $1000?
- What is the minimum number of sections your wheel will need?

Clue: write P(win $1000) as a fraction.

Part 2:

There will be 26 programs in the game show's first season.
In each show, the wheel is spun a total of 15 times.

How much should the producers expect to pay out in prizes?

Things to think about:

- How many times would you expect each event to occur?

Extension

To increase the excitement, one "$100" section is made into a "Bankrupt" section.
If a contestant spins "Bankrupt," they leave, and lose all the prizes they had won to that point.
If they don't spin "Bankrupt," they can either spin again, or keep their prizes and leave.

- What is the probability of landing on the "Bankrupt" section on your wheel?
- What is the probability that a contestant will land on "Bankrupt" in 2 spins? In 3 spins?

Open-ended Extensions

1) The producers want to make the show even more exciting for the viewers.
 - How could you arrange the prizes on your wheel to make it as exciting as possible?
 - How could you change the rules of the game to make the show more exciting?
2) Select a simple card game, game show, board game, or computer game that is based on chance. Describe the game, the outcomes, and the number of ways to obtain each outcome. Calculate the theoretical or experimental probability of each outcome. Does the game try to make it appear that there is a good chance of winning? How?

Round Up

Lots of games are based on chance. That's where probability can be very important. You might not be able to say for certain what will happen, but you can say what's likely.

Chapter 7

Geometry

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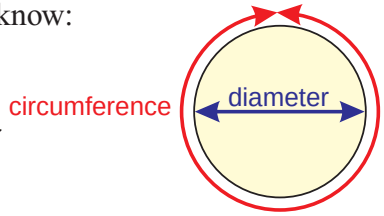
Section 7.1 introduction — an exploration into:

Circle Ratios

Circles are everywhere. For example, wheels are circular. So are CDs. So are some mugs, trashcans, tables, plates, pies... Even the Earth is circular when you look at it from space. It's no wonder you have to learn about them. But there's a lot you can work out for yourself.

First, there are two technical words about circles you need to know:

- The distance all the way around the outside of a circle is called the **circumference**.
- The distance through the center from one side of a circle to the other is called the **diameter**.



Now find **5 circular objects** of **various sizes**. ⇦ For example, a trashcan, a cup, a plate, a CD... **anything**.

You need to measure their **circumference** and **diameter** as **accurately** as possible.

1) Measure the **diameter** of each of your circles using a meter stick or a ruler.
2) Use some **string** to find the **circumference** of each circle.
 - **Loop** the string **around** the the object.
 - **Mark** the length needed to go all the way around.
 - Now **measure** this length, and record the result.
3) Record your measurements in a table like this one.

Object	Diameter (cm)	Circumference (cm)
Mug	9.2	28.9
Plate	24.3	76.6

Example

For each of your circles, find the ratio of the circumference to the diameter.

Solution

Add an extra column to your table.

Divide each circle's **circumference** by its **diameter**.

Record the result as a decimal, to the nearest **hundredth**.

Divide the **bigger** number by the **smaller**.

Use a calculator.

Ratio: circumference ÷ diameter
3.14
3.15

Exercises

1. Is there a pattern to the ratios in the third column of your table?
2. Compare your results with a partner. Are they similar? Or different?
 What about the results of the rest of the class? Are they similar or different?
3. Use your results to complete this sentence:
 For any circle, the ratio of the circumference to the diameter _____.
4. A regulation basketball hoop has a diameter of 18 inches.
 Find its circumference.

 Hint... Use your result from Exercise 3.

Round Up

The link between a circle's circumference and its diameter is one of the most important results in math. And if this Exploration went well, then you've just worked it out for yourself. There's a lot more information about circles in this Section of this book.

Lesson 7.1.1

Section 7.1
Parts of a Circle

Circles are all around you. They're an important type of shape because they have some special features that other shapes just don't have. You're going to learn about these special features in this Lesson.

California Standards:

Algebra and Functions 3.1
Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ — the formulas for the perimeter of a rectangle, the area of a triangle, and the circumference of a circle, respectively).

Algebra and Functions 3.2
Express in symbolic form simple relationships arising from geometry.

What it means for you:
You'll learn about the parts of a circle, including how some of them relate to each other.

Key words:
- circle
- center
- radius
- diameter

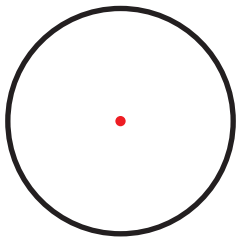
Check it out:
The plural of radius is radii.

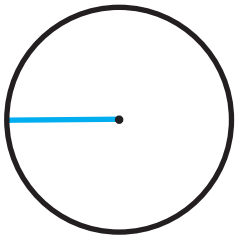
All Circles Have Certain Features in Common

The parts of a circle have **special names**.

The **center** of a circle is the point in the middle that is an equal distance from **all points** of the circle.

The red dot in this picture is at the center.



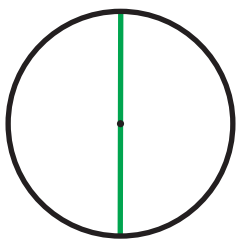


The **radius** of a circle is the **distance** from the center to the edge of the circle. It is shown by a **straight line** from **anywhere** on the edge to the center of the circle.

The blue line in this picture represents the radius of the circle.

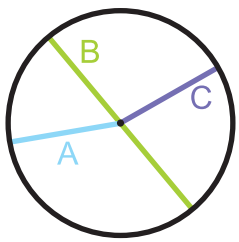
The **diameter** of a circle is the greatest **distance** from one edge of the circle to the other. It is shown by a **straight line** that passes through the center.

The green line in this picture represents the diameter of the circle.



Example 1

Which letters show:
- a radius of this circle?
- a diameter?



Solution

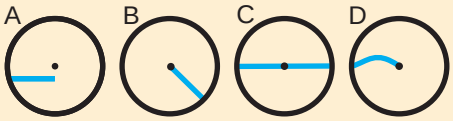
Lines **A** and **C** are both **radii** of the circle.

Line **B** is a diameter of the circle.

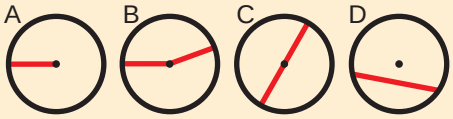
Notice that **half** of line B, from either edge to the center, would also be a radius.

Guided Practice

1. On which of these pictures does the blue line show the radius? Why do the others not?



2. On which of these pictures does the red line show the diameter? Why do the others not?



3. Mrs. Batalles is buying a circular tablecloth for her kitchen table. The tabletop is circular with a diameter of 60 in. She wants the tablecloth to hang down over the table about another 3 in. all around. Which of these tablecloths should she buy? Explain your answer.

a. 66 in. diameter b. 180 in. diameter
c. 63 in. diameter d. 33 in. diameter

4. This figure shows concentric circles. Use the diagram to complete this definition:



Concentric circles are circles that share the same _____.

The Diameter of a Circle Is Double Its Radius

The **diameter** of a circle is **twice** its **radius**.

Example 2

A circle has a radius of 8 inches. What is its diameter?

Solution

The diameter is **twice** the radius. So the **diameter** of the circle is

$$8 \times 2 = \textbf{16 inches}$$

In math, the letter r is often used to represent the radius.

And d is used for the diameter.

So you can write "the radius is 5 cm" as "$r = 5$ cm," and "the diameter is 10 cm" as "$d = 10$ cm."

"The diameter is 2 times the radius" can be written as the formula

$$d = 2r$$

Don't forget:
The letters r and d are variables in this formula. That means they're standing in for numbers you don't know yet.

Example 3

A circle has a diameter of 22 feet. What is its radius?

Solution

You can use the formula $d = 2r$ here.

$d = 2r$ — **Write out the formula**

$22 = 2r$ — **Substitute $d = 22$**

$r = 22 \div 2$ — **Divide by 2**

$r = 11$ **feet**

Guided Practice

5. A circle has diameter 40 inches. What is its radius?

6. A circle has radius 7 cm. What is its diameter?

7. A circle has radius 3 feet. What is its diameter?

8. A circle has diameter 25 inches. What is its radius?

In Exercises 9–14, find the missing measure:

9. $r = 30$ cm, $d = ?$

10. $d = 30$ in., $r = ?$

11. $r = 4.6$ m, $d = ?$

12. $d = \frac{2}{3}$ yd, $r = ?$

13. $d = 6.7$ cm, $r = ?$

14. $r = \frac{2}{3}$ ft, $d = ?$

Independent Practice

1. A circle has radius 17 cm. What is its diameter?

2. A circle has diameter 9 yards. What is its radius?

In Exercises 3–6, find the missing measure:

3. $r = 49$ ft, $d = ?$

4. $r = 11.5$ m, $d = ?$

5. $d = 63$ in., $r = ?$

6. $d = \frac{4}{9}$ yd, $r = ?$

7. Mr. Rodriguez has a circular clock on his wall. The minute hand reaches exactly from the center of the face to the edge, and is 6 inches long. What is the diameter of the clock face?

8. Russell is standing at the edge of a circular lake. Looking straight across the center of the lake, he sees his friend Salvador on the other side. They are 35 yards apart. What is the radius of the lake?

Now try these:

Lesson 7.1.1 additional questions — p462

Round Up

The radius and diameter are really important, because we can use them to work out things like the area and circumference of a circle. Make sure you know what each one means, because you will see them again soon.

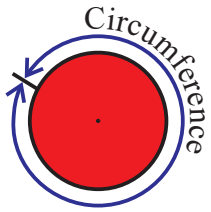
Lesson 7.1.2

Circumference and π

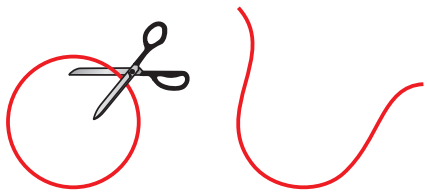
In this Lesson, you'll learn about the distance around the outside of a circle, which is called the circumference. There is a clever way to work out the circumference if you know the radius or diameter of the circle, and it involves a special number.

The Circumference Is Another Important Measurement

The **distance** around the **edge** of a circle is called the **circumference**.



Imagine that a circle is like a loop of string. If you cut the string and laid it out flat, you could measure the **length** of the string.





If you could cut a circle and lay it out in the same way, the length you measured would be the **circumference**.

Example 1

Norman is walking around the edge of a large, circular rug. By the time he is one-third of the way around it, he has walked 8 feet. What is the circumference of the rug?

Solution

The circumference is the distance **all the way around** the rug.

Norman has walked one-third of the way around, so the circumference is 3 times the distance he has walked.

$$3 \times 8 = \mathbf{24}$$

The circumference of the rug is **24 feet**.

Guided Practice

1. Jack is running around a circular track. When he is halfway around, he has run 150 yards. What is the circumference of the track?

2. Judy's scarf is 60 cm long. The scarf will go around a telephone pole exactly 3 times. What is the circumference of the pole?

3. Jorge is riding his bicycle. When he travels 5 m, his front wheel goes around 5 times. What is the circumference of the wheel?

California Standards:

Algebra and Functions 3.1
Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ **— the formulas for the** perimeter of a rectangle, the area of a triangle, and the **circumference of a circle,** respectively).

Algebra and Functions 3.2
Express in symbolic form simple relationships arising from geometry.

Measurement and Geometry 1.1
Understand the concept of a constant such as π; know the formulas for the circumference and area **of a circle**.

Measurement and Geometry 1.2
Know common estimates of π (3.14; 22/7) and use these values to estimate and calculate the circumference and the area **of circles**; compare with actual measurements.

What it means for you:

You'll learn about a special number which you can use in a formula to find the circumference of a circle.

Key words:

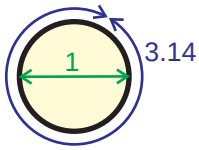
- circumference
- diameter
- radius
- pi (π)
- estimate

Circumference Is Related to Diameter

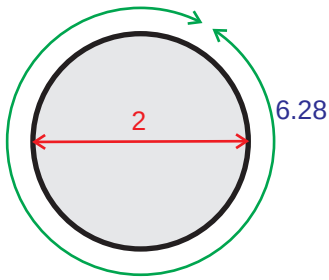
You've seen that the diameter is always twice the radius.

Diameter and circumference are related too — the **circumference** of a circle is always just over **three times** its **diameter**.

The diameter of this circle is **1**.
Its circumference is about **3.14**.



If the diameter is doubled, the circumference also doubles.



In fact, **circumference ÷ diameter** for **any** circle is always **3.1415926535897...** This number is **very important** in math. The full version of it is a **never-ending decimal**.

There's a special symbol mathematicians use instead of writing out the full number. The symbol is a Greek letter called **pi** (pronounced "pie"). It looks like this:

$$\pi$$

π is kind of tricky — it's a **symbol**, so it looks like a **variable**. This is one time where a symbol **doesn't** stand for an unknown. It's just the **name** of a **special number**.

You Can Use π in a Formula to Find Circumference

Using ***C*** for circumference and ***d*** for diameter, you can use **π** to make a **formula** for the circumference of a circle:

$$C = \pi d$$

Check it out:
These formulas are the same because $d = 2r$,
so $\pi \times d = \pi \times 2r$
$= 2 \times \pi \times r$

This formula is often written using ***r*** for radius, like this:

$$C = 2\pi r$$

Example 2

A circle has a diameter of 12 inches. What is its exact circumference?

Solution

You can use the formula $C = \pi d$ to solve this question.

$$C = \pi \times 12$$
$$= 12\pi \text{ in.}$$

You can leave π in an exact answer, because it is a **number**.

Guided Practice

In Exercises 4–9, find the exact circumference of a circle with the given diameter or radius.

4. $d = 20$ in.
5. $d = 13$ ft.
6. $d = 3.5$ m
7. $r = 5$ ft.
8. $r = 26$ cm
9. $r = 6.3$ cm

Don't forget:
To get an exact answer, you'll need to leave π in your answer.

You Can Often Use a Rounded Value of π

If an exercise asks you to give the **exact** circumference, you should always leave π in the answer.

Numbers with π in them are not very easy to work with, so it is more usual to **estimate** the circumference by using a rounded value of π.

If the radius or diameter is given as a **decimal**, you should use **3.14** or **3.142** as the value of π.

If the question uses **fractions**, you should use $\frac{22}{7}$.

You should normally use a rounded value of π in **real-life** situations.

Check it out:
If you need to use an estimate of π in a question, you will usually be told what value you should use.

Example 3

Alicia has a circular swimming pool. The diameter of the pool is 30 ft. What is the circumference of Alicia's pool?

Solution

Using the formula $C = \pi d$, and estimating π as **3.14**, you can solve the question:

$$C = 3.14 \times 30$$
$$= 94.2$$

The circumference of Alicia's pool is **94.2 ft**.

Example 4

T.J. is cutting a circle of paper to use on a poster for a math project. The circle has a radius of $3\frac{1}{4}$ in. What is its circumference?

Solution

You can use the formula $C = 2\pi r$, and estimate π as $\frac{22}{7}$:

$$\begin{aligned} C &= 2 \times \frac{22}{7} \times 3\frac{1}{4} \\ &= 2 \times \frac{22}{7} \times \frac{13}{4} \\ &= \frac{572}{28} = \frac{143}{7} \\ &= 20\ \frac{3}{7} \end{aligned}$$

So the circumference of the circle is $20\ \frac{3}{7}$ **in.**

Check it out:

Be careful when you are answering circumference questions. Make sure that you check whether the question gives the radius or the diameter.

Guided Practice

In Exercises 10–21, calculate the circumference of a circle with the given diameter or radius. Use $\pi = 3.14$ for Exercises 10–15.

10. $d = 15$ cm **11.** $r = 11$ yd **12.** $d = 42$ in.

13. $r = 6$ m **14.** $d = 9.2$ ft **15.** $r = 13.5$ m

Use $\pi = \frac{22}{7}$ for Exercises 16–21.

16. $r = \frac{1}{3}$ yd **17.** $d = \frac{5}{8}$ m **18.** $r = \frac{4}{11}$ in.

19. $r = 12\frac{3}{7}$ m **20.** $d = 9\frac{1}{9}$ ft **21.** $d = 1\frac{5}{17}$ ft

Independent Practice

In Exercises 1–9, use a suitable estimate of π to find the circumference of a circle with the given diameter or radius:

1. $r = \frac{5}{6}$ ft **2.** $r = 2.75$ cm **3.** $d = 6.3$ m

4. $d = 13$ in. **5.** $r = \frac{7}{11}$ m **6.** $d = 0.75$ yd

7. $r = 3.3$ ft **8.** $d = \frac{4}{3}$ m **9.** $r = 8.1$ in.

10. Complete the sentence by filling in the blanks.
π is the ratio of a circle's c _ _ _ _ _ _ _ _ _ _ _ _ to its d _ _ _ _ _ _ _.

11. A good estimate of the height of an elephant from its foot to its shoulder is twice the circumference of its (roughly circular) foot. A zoologist in Africa finds an elephant footprint with a 12-inch diameter. Estimate how tall the elephant that left the footprint is.

Now try these:

Lesson 7.1.2 additional questions — p462

Round Up

π is a really useful and important number. It might look a little strange, but remember, it is just a fixed number. You will see π again many times in math, including in the next Lesson of this book.

Lesson 7.1.3

Area of a Circle

You've already worked out the circumference of a circle using π. You might need to find the area of a circle too.

There are a couple of ways of working out the area of a circle. You can either estimate it, or you can work it out using a formula that uses the radius and a value of π. That's what this Lesson is about.

California Standards:

Measurement and Geometry 1.1
Understand the concept of a constant such as π; **know the formulas for the** circumference and **area of a circle**.

Measurement and Geometry 1.2
Know common estimates of π (3.14; 22/7) and use these values to estimate and calculate the circumference and the **area of circles**; compare with actual measurements.

What it means for you:
You'll learn a formula that will help you find the area of a circle. You'll use the formula with exact and estimated values of π.

Key words:
- area
- radius
- diameter
- pi (π)
- estimate

Don't forget:
One square unit is the amount of space taken up by a square with a side length of one unit.

The Area of a Circle Is Measured in Square Units

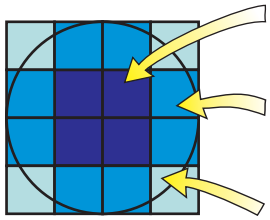
The area of a circle refers to the number of **square units** inside the circle.

Example 1

Estimate the area of a circle with a radius of 2.

Solution

Draw the circle on grid paper and **count** the squares inside the circle.



There are **4** squares **completely** inside the circle.



There are **8** squares **almost completely** inside the circle.



There are **4** squares that are about **one-quarter** inside the circle.



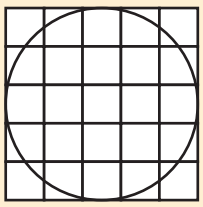
Estimated area $= 4 + 8 + (4 \times \frac{1}{4}) = 13$.

So the area is about **13 square units**.

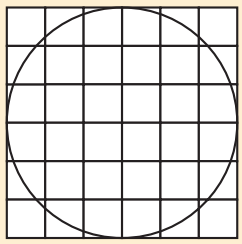
Guided Practice

In Exercises 1–2, find the area of each circle by counting squares.

1.



2.



In Exercises 3–6, find the area of each circle by drawing the circle and counting squares.

3. $r = 6$ **4.** $r = 4$ **5.** $r = 3.5$ **6.** $d = 10$

You Can Find the Area of a Circle Using a Formula

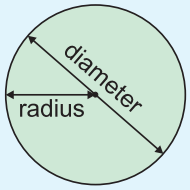
Counting squares is okay if the circle is small, but not if the circle is large. There is a formula you can use to find the area of a circle.

The formula for the area of a circle is:

$$A = \pi r^2$$

where r is the radius of the circle.

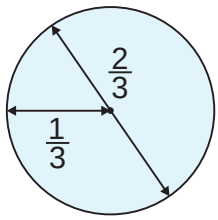
Don't forget:



For more about parts of a circle, see Lesson 7.1.1

Example 2

Find the area of a circle with a diameter of $\frac{2}{3}$ mile.



Solution

First find the radius. $r = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

Since the radius is a fraction, use a fraction estimate for π.

$$A = \pi r^2$$

$$= \frac{22}{7}\left(\frac{1}{3}\right)^2 \quad \text{Substitute } \frac{1}{3} \text{ for r and substitute } \frac{22}{7} \text{ for } \pi.$$

$$= \frac{22}{7} \times \frac{1}{9} \quad \text{Simplify exponents first.}$$

$$= \frac{22}{63} \quad \text{Then multiply out.}$$

The area of the circle is approximately $\frac{22}{63}$ **square miles**.

Don't forget:

If you use an estimated version of π (such as 3.14 or $\frac{22}{7}$), then the answer is only an estimate too.

If an exercise asks for an exact area, you just need to leave π in the answer.

Guided Practice

In Exercises 7–15, find the **estimated** area of each circle with the given radius or diameter. Use π = 3.14.

7. $r = 11$ in. **8.** $r = 2.1$ cm **9.** $d = 14$ yd
10. $d = 20$ m **11.** $r = 6.4$ m **12.** $d = 42.2$ cm
13. $d = 16.1$ yd **14.** $d = 40$ m **15.** $d = 86.4$ m

In Exercises 16–24, find the **exact** area of each circle with the given radius or diameter.

16. $r = 8$ in. **17.** $r = 7.1$ cm **18.** $d = 44$ yd
19. $d = 19$ m **20.** $r = 9$ m **21.** $d = 21.4$ cm
22. $d = 12.5$ yd **23.** $d = 16.4$ m **24.** $d = 99.6$ m

Use a Rounded Value of π in Real-Life Situations

When you're applying the area formula in real-life situations, you usually need to use a rounded value of π.

Example 3

A weed-control company charges $1.25 a square foot to spray for weeds. If a circular flower garden has a radius of 7 ft, how much will the company charge to spray it for weeds?

Solution

$A = \pi r^2 = 3.14 \times r^2$ **First substitute π for a rounded value.**

$= 3.14 \times 7^2$ **Substitute 7 for *r*.**

$= 3.14 \times 49$ **Simplify all exponents.**

$= 153.86$

The approximate area is **153.86 ft²**.

Now work out the cost:

Cost = (Area in square feet) × (Cost per square foot)

$= 153.86 \times 1.25 = 192.325$.

Rounded to the nearest cent, the company will charge **$192.33**.

Guided Practice

25. A water sprinkler can be set to rotate in a full circle. If the radius of the spray is 5 ft, how many square feet of grass will the sprinkler cover in one full rotation?

26. A dessert plate has a radius of 3 in. A dinner plate has a radius of 5 in. How many more square inches of area does a dinner plate have?

Independent Practice

In Exercises 1–5, find the estimated area of each circle. Use π = 3.14.

1. $r = 8$ in. **2.** $r = 3.2$ cm **3.** $d = 12$ yd **4.** $d = 18$ m **5.** $r = 5.8$ m

6. Paint is purchased in 1-quart cans. A quart of paint covers between 110 and 125 square feet. If you need to paint four circles that are 6 feet in diameter, how many cans of paint should you buy?

7. If you order two 12-in.-diameter pizzas, is this the same amount of pizza as one 24-in.-diameter pizza? Explain your answer.

8. Write a formula that could be used to find the area of a semicircle.

Now try these:
Lesson 7.1.3 additional questions — p463

Check it out:
A semicircle is half a circle:



Round Up

This is another example of how useful π is. Using the formula makes calculating the area of a circle a whole lot easier than if you tried to count squares on graph paper. Later in this Chapter, you'll use the circle area formula as a first step in calculating the volume of a cylinder.

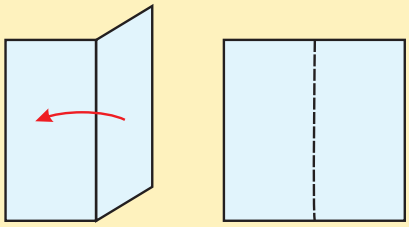
Section 7.2 introduction — an exploration into:

Folding Angles

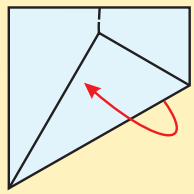
This Exploration is all about angles and triangles. The first part of the Exploration involves making all the angles and triangles you're going to need by folding a sheet of paper. Do this as carefully as you can or the second part of the Exploration will become much more difficult.

For this exploration you'll need a **square** sheet of paper.
You need to **fold** your sheet of paper by **carefully** following the directions below.

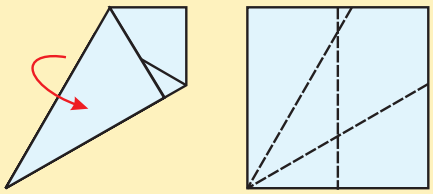
1) Fold the paper **in half** lengthwise. Unfold the paper.



2) Fold the **bottom-right corner** of the square to the **center line** — with the fold going through the **bottom-left corner**.



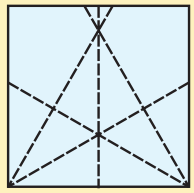
3) Fold the **left edge** of the paper down, so that it meets the **other folded edge**. Then unfold your paper.



4) **Repeat steps 2 and 3**, but this time fold the **bottom-left corner** to the middle, then the **right edge** over.

Now unfold your paper, and make **pencil lines** along each fold.

Use a **ruler**.

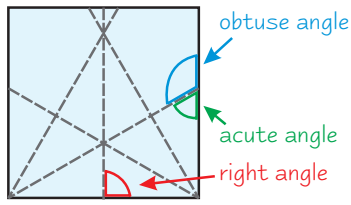


Example

Identify one of each of the following types of angle:
• right (= 90°), • acute (less than 90°), • obtuse (greater than 90°).

Solution

See the diagram.



Exercises

1. What is an equilateral triangle? Identify any equilateral triangles formed by the folds.
2. Identify as many types of triangles and other shapes formed by the folds as you can.
3. Identify the measures of as many angles formed by the folds as you can. Say whether each angle is acute, obtuse, or right. Justify your answers.

Work them out. **Don't** use a protractor.

Round Up

You'll see lots about angles and their properties in this Section. Once you've studied the Section, look back at this Exploration and see if you can identify any special pairs of angles.

Lesson 7.2.1

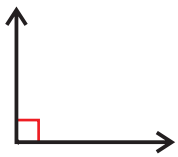
Section 7.2 Describing Angles

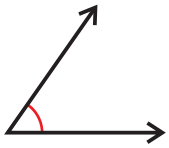
Whenever two straight lines meet at a point, there is an angle between them. This Lesson is about describing angles by naming the different types and by measuring them.

Angles Are Measured in Degrees

You can measure **how large** the angle between two straight lines is. The angle is measured at the point where the lines meet. The units used to measure an angle are called **degrees**. The symbol ° means degrees.

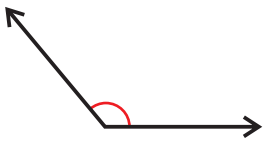
A **right angle** measures **exactly 90°** and looks like the corner of a square or rectangle.





An angle measuring between 0° and 90° is called an **acute angle**.

An angle measuring between 90° and 180° is called an **obtuse angle**.

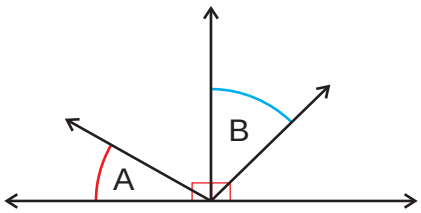


Because you know exactly what a 90° right angle looks like, you can use it to **estimate** the measure of other angles.

Example 1

Estimate the sizes of angles A and B shown on the right.

Are theses angles acute, obtuse, or right angles?



Solution

The small squares on the picture show where there are **right angles**, which you know are **90°**.
You can use this to estimate angles A and B.

It looks like angle A is about **one-third** as big as the right angle shown on the left. So 90 ÷ 3 = **30°** might be a good estimate for angle A.

Angle B looks like it is about **half** the other right angle shown.
So a good estimate of the size of angle B might be 90 ÷ 2 = **45°**.

Both angles are **less than 90°**, so they are both **acute**.

California Standards:

Measurement and Geometry 2.1
Identify angles as vertical, adjacent, complementary, or supplementary **and provide descriptions** of these terms.

Measurement and Geometry 2.2
Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to **solve problems involving an unknown angle**.

What it means for you:

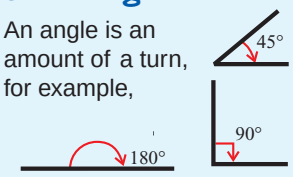
You'll learn about angles – some names for different types, how to measure them, and how to draw them.

Key words:

- angle
- acute
- right angle
- obtuse
- protractor
- ray
- vertex

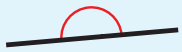
Don't forget:

An angle is an amount of a turn, for example,



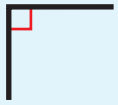
Check it out:

An angle that measures 180° would be a straight line.



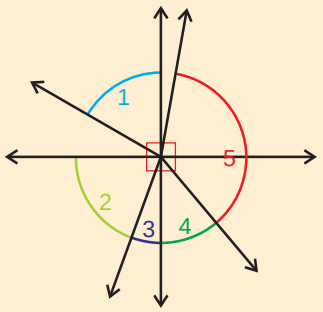
Check it out:

Right angles are usually marked by a little square at the corner.



Guided Practice

Use the diagram to answer Exercises 1–6:



1. Which of the angles shown are acute and which are obtuse?

2. Estimate the size of angle 1.

3. Estimate the size of angle 3.

4. Estimate the size of angle 4.

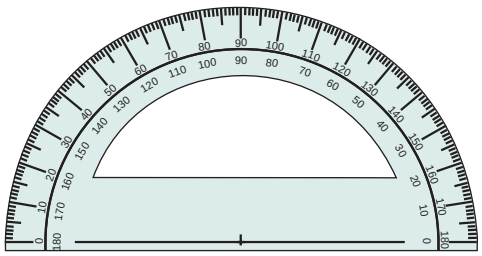
5. Esteban estimates that the size of angle 2 is about 45°. Is this a reasonable estimate? Give a reason for your answer.

6. Abigail estimates that the size of angle 5 is about 175°. Is this a reasonable estimate? Give a reason for your answer.

A Protractor Can Be Used to Measure Angles

Check it out:
There are usually two sets of numbers around the edge of a protractor. Make sure you go for the one that makes sense, by thinking about whether the angle is more or less than 90°.

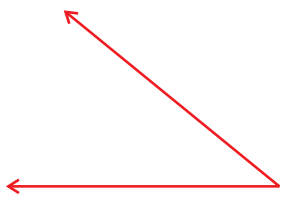
A **protractor** is a tool you can use to **measure** the size of an angle. The protractor has 180 small marks around its edge, each representing an angle of 1°.



Example 2

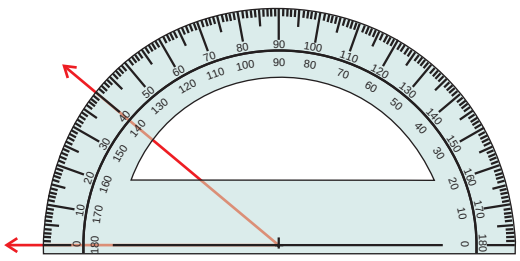
Check it out:
The lines that form an angle can also be called the rays of the angle.
The vertex is another name for the point where the rays meet.

Use a protractor to measure the size of this angle.



Solution

Place the bottom line on the protractor **directly over** one of the rays of the angle. The point marking the **center** should be above the vertex of the angle.



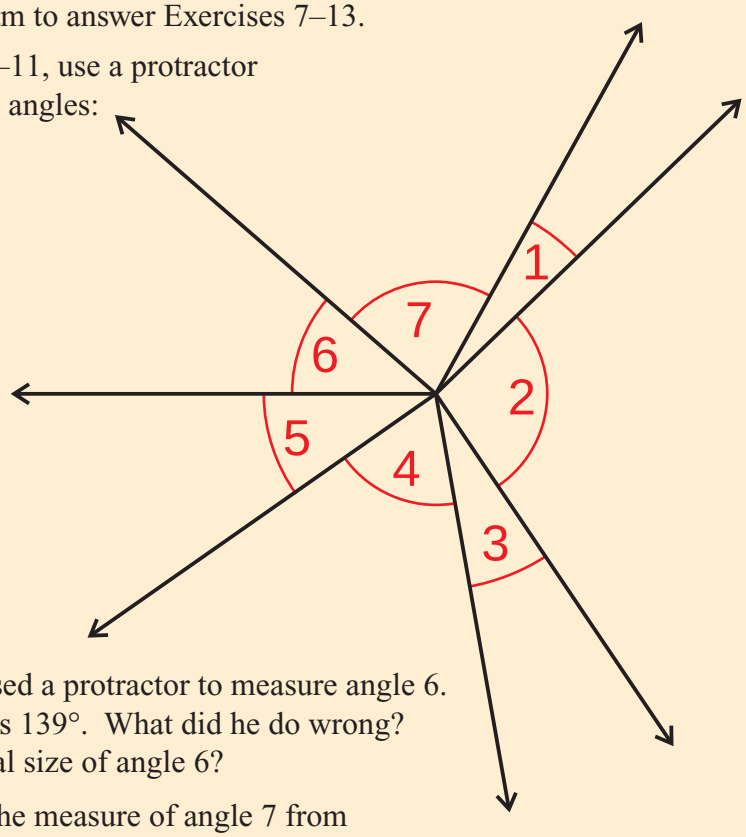
You **read the angle** at the point where the second ray crosses the **curved edge** of the protractor. This angle measures **40°**.

Guided Practice

Use the diagram to answer Exercises 7–13.

In Exercises 7–11, use a protractor to measure the angles:

7. Angle 1
8. Angle 2
9. Angle 3
10. Angle 4
11. Angle 5



12. Michael used a protractor to measure angle 6. His answer was 139°. What did he do wrong? What is the real size of angle 6?

13. Work out the measure of angle 7 from the values of angles 1–6, then check your answer with a protractor.

Check it out:
The total angle measure going all the way around one point is 360°.

A Protractor Can Also Be Used to Draw Angles

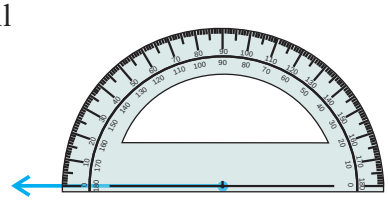
You can use a protractor for **drawing** angles of a **given size**.

Example 3

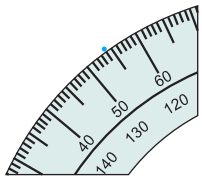
Draw an angle that measures 54°.

Solution

Start by drawing a **straight line**. This will be one of the rays that form the angle.

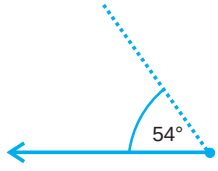


Line it up with the bottom line on the protractor, with one end of the ray at the center mark of the protractor.



Follow the scale around the protractor until you get to 54°. **Make a mark** at the 54° point.

Finally, remove the protractor and **join the mark up** with the end of your original ray.



Guided Practice

Use a protractor to draw the following angles:

14. 80° **15.** 27° **16.** 99°
17. 165° **18.** 49° **19.** 133°

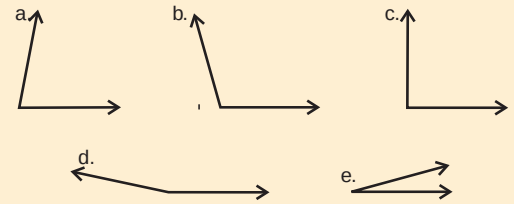
Independent Practice

Exercises 1–6 are about hands on a clock. For each time, say whether the angle made by the clock hands will be acute, obtuse, or right.

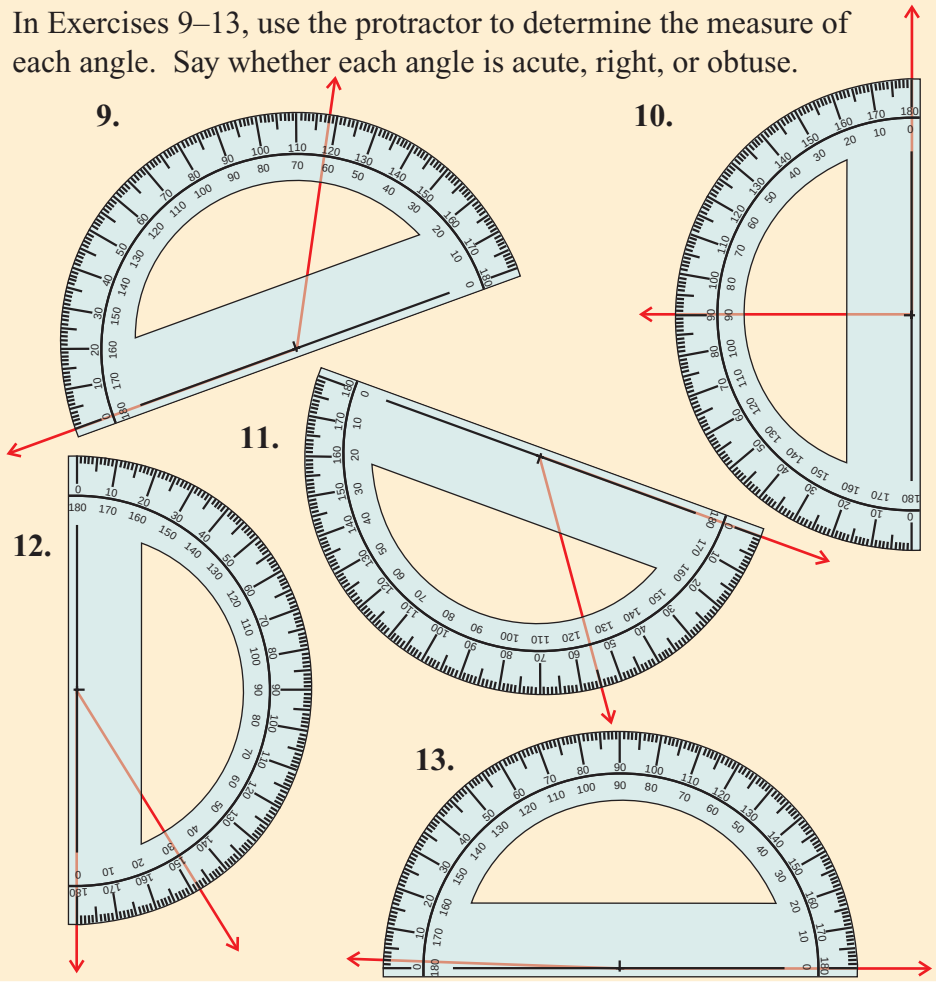
1. 3:00 **2.** 11:00 **3.** 1:15
4. 9:26 **5.** 7:30 **6.** 6:55

7. When cutting paper with a pair of scissors, is the angle between the blades of the scissors usually acute, obtuse, or right? Explain why.

8. Without measuring, match each of these angles with its measure. Choose from 15°, 80°, 90°, 105°, or 168°.



In Exercises 9–13, use the protractor to determine the measure of each angle. Say whether each angle is acute, right, or obtuse.



Now try these:
Lesson 7.2.1 additional questions — p463

Round Up

Angles are useful for describing the different shapes you'll learn about later in this Chapter. You should be able to picture how big an angle is when you are told its size in degrees.

Lesson 7.2.2

Pairs of Angles

California Standard:
Measurement and Geometry 2.1
Identify angles as vertical, adjacent, complementary, or supplementary **and provide descriptions of these terms.**

What it means for you:
You're going to learn about some special types of angles.

Key words:
- angle
- adjacent angles
- linear pair
- vertical angles
- ray
- vertex

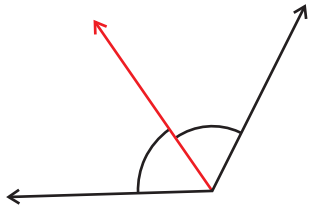
Don't forget:
The rays are the sides of the angle. The vertex is the point where the rays meet.

In the last Lesson, you learned about different types of angles. Sometimes angles come in pairs. In this Lesson, you'll meet three special types of angle pairs.

Adjacent Angles Share One Ray and Don't Overlap

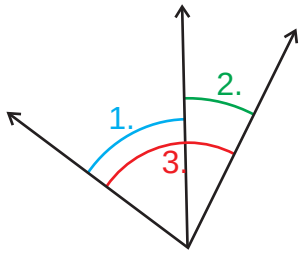
A pair of angles are said to be **adjacent** if they are found **next to each other** on either side of one ray. No parts of the angles **overlap** each other.

The two angles shown here are **adjacent**. The ray marked in red is a ray of **both angles**, but no parts of the angles themselves are the same.



Example 1

In this diagram:
- Is angle 1 adjacent to angle 2?
- Is angle 1 adjacent to angle 3?



Solution

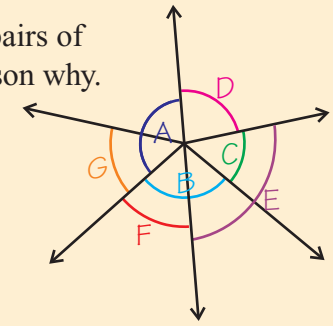
The middle ray forms a side of both **angle 1** and **angle 2**. They have **one ray** in common and **don't overlap**, so angles 1 and 2 **are** adjacent.

Angles 1 and 3 share **one ray**, on the left-hand side of the diagram. But they **aren't** adjacent, because they overlap each other.

Guided Practice

In Exercises 1–10, say whether the following pairs of angles are adjacent. If they are not, give a reason why.

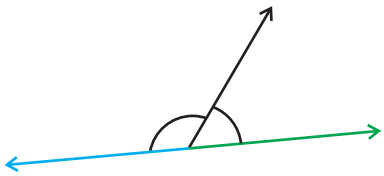
1. A and B
2. A and C
3. C and E
4. C and F
5. D and E
6. A and G
7. B and E
8. B and G
9. D and F
10. D and G



A Linear Pair Is a Special Pair of Adjacent Angles

In a pair of **adjacent angles**, there are two "outside" rays that are part of only **one angle each**. When these two rays form a **straight line**, the pair of angles is called a **linear pair**.

Check it out:
The measures of a linear pair of angles always add up to 180°.



In this picture, the **blue ray** is part of the angle on the **left**, and the **green ray** is part of the angle on the **right**.

Together these rays form a **straight line**, so the pair of angles is a **linear pair**.

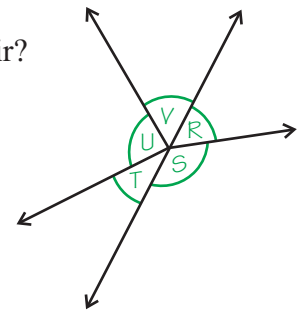
Example 2

Which two angles in this diagram are a linear pair?

Solution

Only two rays form a **straight line**. The two angles formed by **one shared ray** and two other rays that make the **straight line** are R and S.

So angles **R and S** form a linear pair.



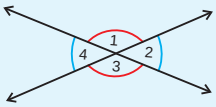
Guided Practice

For Exercises 11–12, use the diagram in the previous Guided Practice.

11. Which one of the following pairs of angles is a linear pair:
A and B A and D A and F

12. Give another pair of angles on the diagram that are a linear pair.

Check it out:
Vertical angles are equal, because...

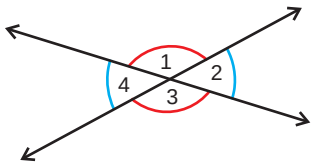


... angles 1 and 2 are a linear pair
... angles 1 and 4 are also a linear pair.
The measures of a linear pair of angles always add up to 180°, so angles 2 and 4 must have the same measure.

Vertical Angles Share the Same Vertex

Vertical angles are formed when **two straight lines cross**. The rays of one angle are **opposite** the rays of the other angle. A pair of vertical angles share the **same vertex** and have the **same size**.
In this diagram:

- Angles **1 and 3** are vertical angles.
- Angles **2 and 4** are also vertical angles.



Example 3

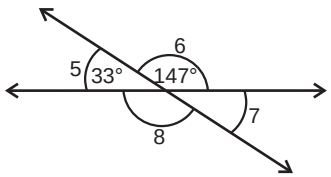
On the diagram below, identify:

- the vertical angles
- the linear pairs

Find the measures of angles 7 and 8.

Solution

Angles **5 and 7** are vertical angles.
Angles **6 and 8** are also vertical angles.



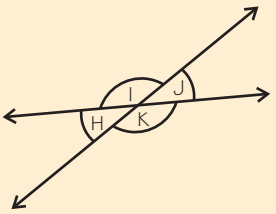
The linear pairs are:
Angles **5 and 6**, angles **5 and 8**, angles **6 and 7**, and angles **7 and 8**.

Vertical angles always have the same measure, so angle 7 measures **33°** and angle 8 measures **147°**.

Guided Practice

Use the diagram to answer Exercises 13–15.

13. Identify the vertical angles in the diagram.



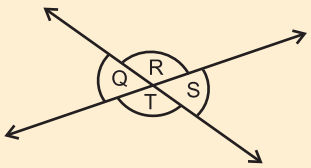
Angle H measures 28° and angle I measures 152°.

14. Find the size of angle J.

15. What is the size of angle K?

Use the diagram below to answer Exercises 16–19.

16. Identify the linear pairs and vertical angles in the diagram.



Angle S measures 55°.

17. Work out the size of angle Q.

18. Find the size of angle R.

19. Find the size of angle T.

Independent Practice

In Exercises 1–4, say whether the statements are true or false. If false, explain why.

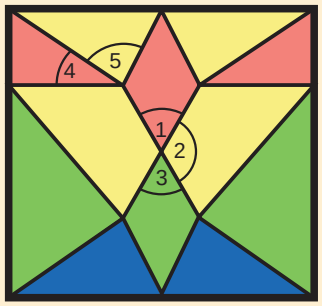
1. A linear pair of angles always forms a straight line.

2. Two adjacent angles can never have equal measures.

3. The angles forming a linear pair are never adjacent.

4. Two adjacent angles can also be vertical angles.

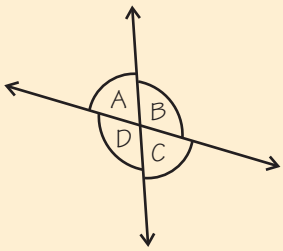
In Exercises 5–7, use the numbered angles on this stained-glass window design to identify each of the following:



5. a pair of vertical angles

6. a linear pair of angles

7. a pair of adjacent angles that are not a linear pair



8. Gordon and Ana are looking at this diagram. Gordon claims it is a sketch of vertical angles. Ana says it is a sketch of adjacent angles. Explain why both students are correct.

Now try these:
Lesson 7.2.2 additional questions — p464

Round Up

You should make sure you know these three types of angle pairs pretty well. Adjacent angles and linear pairs will be particularly important in the rest of this Section.

Lesson 7.2.3

Supplementary Angles

You don't always have to measure with a protractor to find the size of an angle. There are times when you can figure out the size of one of a pair of angles if you know the size of the other.

California Standards:

Measurement and Geometry 2.1

Identify angles as vertical, adjacent, complementary, or **supplementary and provide descriptions of these terms.**

Measurement and Geometry 2.2

Use the properties of complementary and **supplementary angles** and the sum of the angles of a triangle **to solve problems involving an unknown angle.**

What it means for you:

You'll meet another special type of angle pair, and see how this can be useful for solving problems.

Key words:

- angle
- supplementary angles
- supplement
- linear pair

Don't forget:

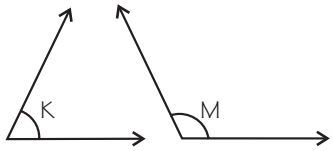
For more about equations, see Section 2.2.

Supplementary Angles Add Up to 180°

Any two angles with measures that **add up to 180°** are called **supplementary angles.**

Example 1

Angle K measures 65°. Angle M measures 115°. Are angles K and M supplementary?



Solution

You can **add the sizes** of two angles together to see if they are supplementary.

$$65° + 115° = \mathbf{180°}$$

So angles K and M **are supplementary**.

When two angles are supplementary, we say that one is the **supplement** of the other. In Example 1, **angle K is the supplement of angle M**.

Example 2

Angle S measures 108°. Find the measure of its supplement.

Solution

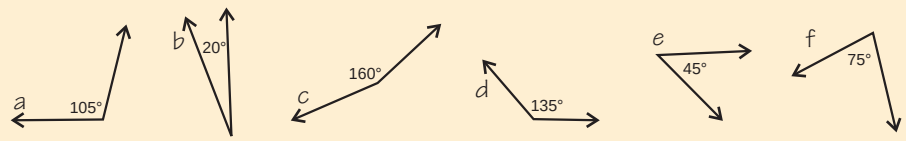
This type of question can be solved using an **equation**.

$108 + n = 180$	**Let n = the measure of the supplement to S**
$n = 180 - 108$	**Subtract 108 from both sides**
$n = 72$	**Solve to find n**

The supplement of angle S measures **72°**.

Guided Practice

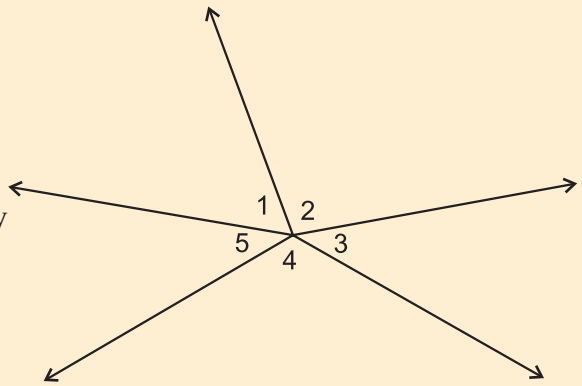
1. Match the angles below into three supplementary pairs.



In Exercises 2–6, find the size of the supplement of an angle with the given measure:

2. 80° **3.** 1° **4.** 65° **5.** 124°
6. 16° **7.** 32°

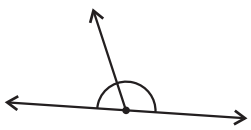
8. Use a protractor to measure the angles on this diagram. Which two angles are a supplementary pair?



Linear Pairs Are Always Supplementary

Any angle with two rays that make a **straight line** measures **180°**.





The angles in a linear pair are on a straight line, so added together they must measure **180°**.

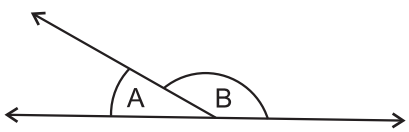
So the angles in a **linear pair** are **always supplementary** to each other.

Check it out:
The fact that an angle on a straight line measures 180° is very useful and important. Make sure you remember it.

Don't forget:
For more about linear pairs, see Lesson 7.2.2.

Example 3

Angle A measures 30°.
Find the size of angle B.



Solution

A and B are a **linear pair** of angles, so together they add up to **180°**.

You can now use an **equation** to find the measure of angle B.

$30 + n = 180$ **Let n = the measure of angle B**

$n = 180 - 30$

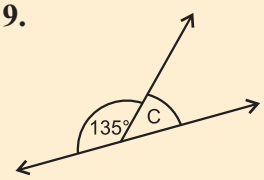
$n = 150$

Angle B measures **150°**.

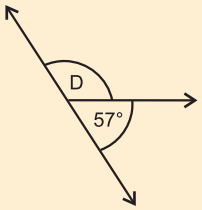
Guided Practice

Find the measure of the missing angles in Exercises 9–11.

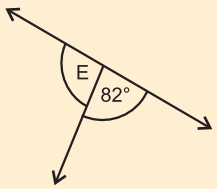
9.



10.



11.

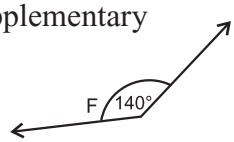


There Are Two Ways to Draw Supplementary Angles

If you're asked to **draw** a supplementary angle, there are **two ways** to do it.

Example 4

Angle F measures 140°. Draw an angle that is supplementary to angle F.

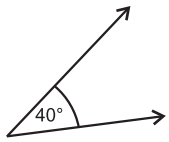


Solution

Method 1:

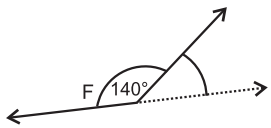
Work out the angle you need using the fact that supplementary angles add up to 180°.

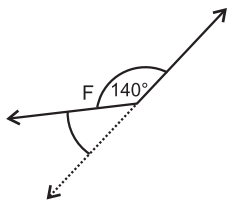
180 – 140 = **40**, so the angle you need to draw is **40°**.
You can draw this with a **protractor**.



Method 2:

You can use a straightedge to **extend** one of the **rays** of angle F. This makes a **linear pair**.
You could create either one of the linear pairs shown.





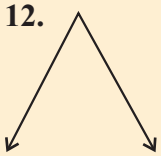
Don't forget:

You'll need to use a protractor to copy these angles. First you'll need to measure how big each one is. Then you'll need to draw an angle that size in your book. See Section 7.2.1 if you've forgotten how to do this. Or you could trace the angles instead.

Guided Practice

In Exercises 12–15, copy the angle shown, then draw its supplement by adding only one more ray to the sketch.

12.



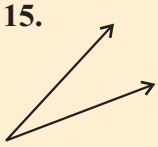
13.



14.



15.

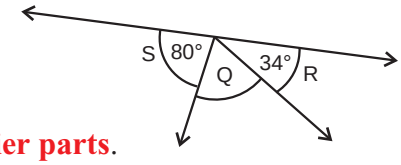


You Can Work Out More Than Two Angles on a Line

Using what you've learned about **supplementary angles** and **linear pairs**, you can solve more complex problems.

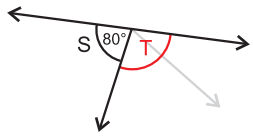
Example 5

Find the measure of angle Q.



Solution

You can break this question into **two easier parts**.
First, imagine **combining** angles Q and R to make a new angle, **T**.



Angles T and S are a **linear pair** so their measures add up to 180°.

So the measure of angle T is 180° – 80° = **100°**.

But you know that angle R + angle Q = angle T.

So $34 + n = 100$ **Let n = the measure of angle Q**

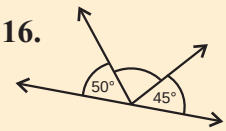
$n = 100 - 34 = 66$

The measure of angle Q is **66°**.

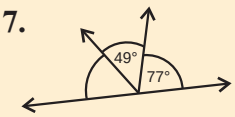
Guided Practice

In Exercises 16–18, calculate the missing angle on each sketch.

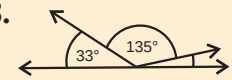
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17.



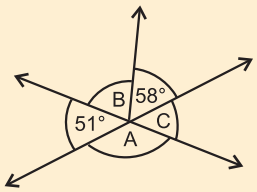
18.



Independent Practice

In Exercises 1–3, use the diagram shown to calculate the measure of:

1. Angle A **2.** Angle B **3.** Angle C



4. Can two acute angles be supplementary? Explain your answer.

5. Can two obtuse angles be supplementary? Explain your answer.

6. Can two right angles be supplementary? Explain your answer.

7. Two angles are supplementary. One angle is three times as large as the other. Find the measure of each angle.

8. Two angles are supplementary. One angle measures 30° more than the other angle. Find the measure of each angle.

Now try these:
Lesson 7.2.3 additional questions — p464

Round Up

Supplementary angles are really useful because you can often use them to find the exact measure of an unknown angle. This is better than using a protractor, because it's easy to make mistakes with a protractor and be a degree or two out.

Lesson 7.2.4

The Triangle Sum

California Standard:

Measurement and Geometry 2.2

Use the properties of complementary and supplementary angles and **the sum of the angles of a triangle to solve problems involving an unknown angle.**

What it means for you:

You'll learn some useful rules about the angles in a triangle.

Key words:

- angle
- triangle sum
- linear pair

Check it out:

Cut a triangle out of paper.



Tear off the three angles



You can place them along a straight line



This shows that the angles' measures add up to 180º.

Check it out:

There are different ways to refer to an angle.

In Example 1, you could say the angle is "the angle at point M."
Or you could call it "the angle LMN" (written ∠LMN), since if you point to points L, M, and N in that order, you trace out the angle at M.

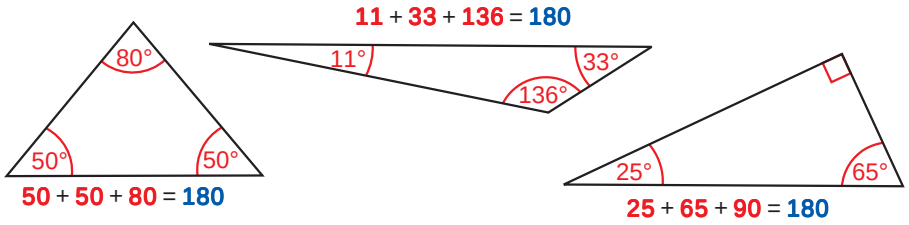
Check it out:

"The triangle with corners L, M, and N" is often written ΔLMN.

So far you've mostly been thinking about angles around a single point. But when you have a drawing of a shape, there's an angle made at each corner. In this Lesson, you'll be learning one of the most useful things you'll ever learn about a triangle.

Angles in a Triangle Add Up to 180°

A **triangle** is made up of **three angles**.
These angles always add up to **180°**.

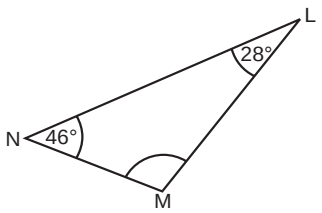


This fact is called the **triangle sum**.

If you have a triangle where one angle is **unknown**, you can use the triangle sum to **work out** the missing angle.

Example 1

Find the measure of the angle at M.



Solution

You could try to find the measure using a **protractor**. But if the diagram isn't accurate, a protractor is useless.

However, you can work out the **exact measure**, because you know that all the angles in a triangle add up to **180°**.

So you can write an **equation** to solve this problem.

$x + 28 + 46 = 180$ **Let x = the measure of angle M**

$x + 74 = 180$

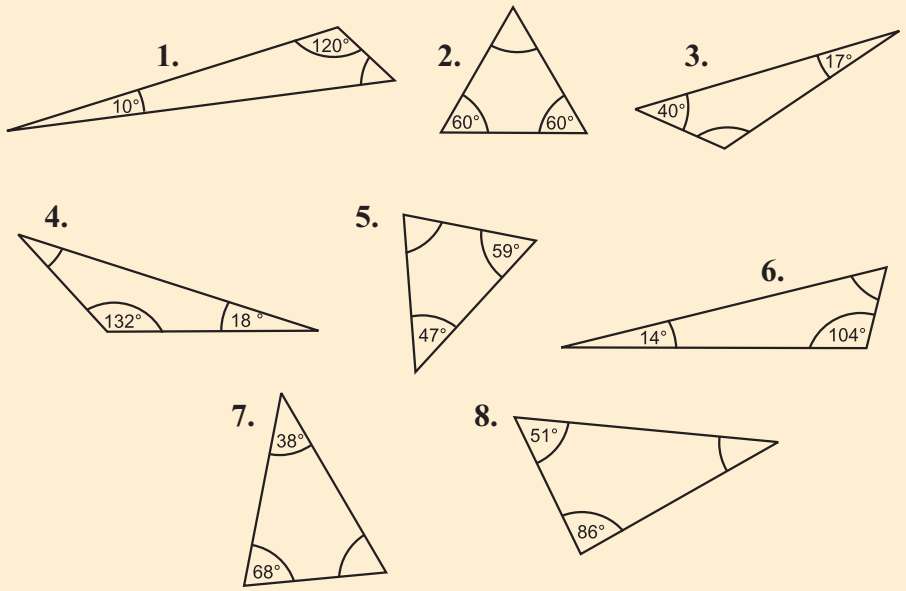
$x = 180 - 74$

$x = 106$

So angle M measures **106°**.

Guided Practice

In Exercises 1–8, find the missing angle on each sketch.



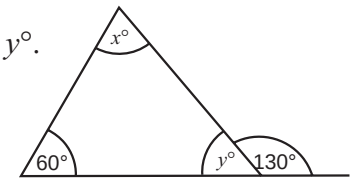
Sometimes You Can Find Two Missing Angles

Sometimes you'll only know the measure of **one angle** inside a triangle. But there might be **other information** you can use to work out the size of the other two angles.

Don't forget:
The angles in a linear pair always sum to 180°.

Example 2

Find the values of the angle measures $x°$ and $y°$.



Solution

The first step is to **find y**. The angle of $y°$ makes a **linear pair** with the 130° angle:

$$y + 130 = 180$$

$$y = 180 - 130$$

$$\mathbf{y = 50}$$

Now you can use the **triangle sum** to find x.

$$x + 60 + 50 = 180$$

$$x + 110 = 180$$

$$x = 180 - 110$$

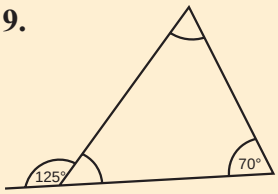
$$\mathbf{x = 70}$$

So $\mathbf{x° = 70°}$ and $\mathbf{y° = 50°}$.

Guided Practice

In Exercises 9–12, find the two missing angles in the triangles shown.

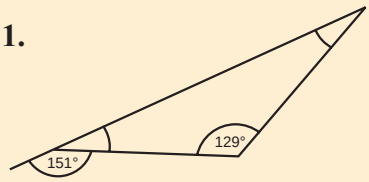
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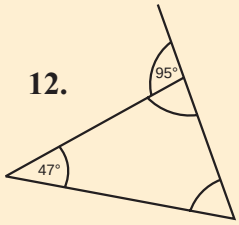
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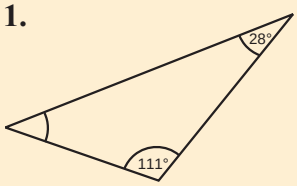
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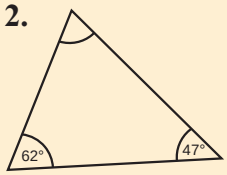
Independent Practice

In Exercises 1–4, find the missing angles.

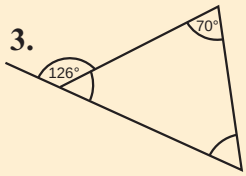
1.



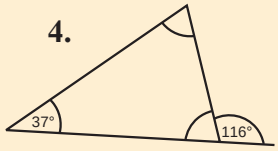
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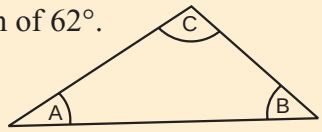
3.



4.



5. The measures of angles A and B have a sum of 62°. Find the measure of angle C.



6. Which three of the following could be the measures of three angles of a triangle: 155° 25° 10° 20° 15° 100°

Exercises 7–8 are about Andrea, who is using her protractor to draw a triangle. She decides to use angles measuring 20°, 85°, and 12°.

7. Explain to Andrea why the measures she has chosen will not work.

8. If she draws a triangle with angles of 20° and 85°, what must the measure of the third angle be?

9. Gil measured the angles of a triangle and recorded them as 40°, 140°, and 100°. Looking at his notebook, he realized that he used the wrong scale on his protractor when measuring one of the angles. Which angle was wrong and how can you tell?

Now try these:
Lesson 7.2.4 additional questions — p464

Round Up

This is just one of many math rules to do with triangles that you'll come across — but it's one of the most important. You'll need to use the triangle sum in many areas of math in the future.

Lesson 7.2.5

Complementary Angles

California Standard:

Measurement and Geometry 2.2

Use the properties of complementary and supplementary **angles** and the sum of the angles of a triangle **to solve problems involving an unknown angle.**

What it means for you:

You'll find out about another special type of angle pair.

Key words:

- angle
- complementary angle
- complement
- right angle

A couple of Lessons ago, you learned all about supplementary angles, which add up to 180°. Complementary angles are very similar, but they add up to 90°.

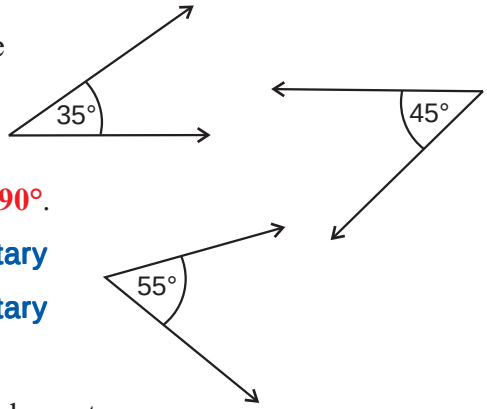
Complementary Angles Add Up to 90°

Any two angles with measures that add up to **90°** are called **complementary angles**.

When two angles are complementary, we can say that each angle is the **complement** of the other.

Example 1

Which two of these three angles are complementary?



Solution

Complementary angles add up to **90°**.

35 + 45 = 80 **Not complementary**

45 + 55 = 100 **Not complementary**

35 + 55 = **90** **Complementary**

So the angles **35° and 55°** are complementary.

Example 2

Angle A is the complement of angle B. Angle B measures 65°. Find the size of angle A.

Solution

You know that complementary angles add up to **90°**.
So you can solve this question using an **equation**.

$n + 65 = 90$ **Let n = the measure of angle A**

$n = 90 - 65$

$n = 25$

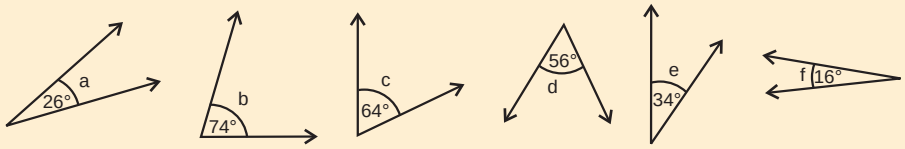
So angle A measures **25°**.

Guided Practice

In Exercises 1–8, find the complement of the given angle.

1. 60° **2.** 11° **3.** 27° **4.** 37°
5. 84° **6.** 59° **7.** 41° **8.** 76°

9. Make three complementary pairs from the angles below.



Adjacent Complementary Angles Make a Right Angle

You know that complementary angles have a sum of **90°**, and you also know that an angle that measures 90° is a **right angle**.

So if you put a pair of complementary angles together to make an **adjacent pair**, the two angles together form a right angle.

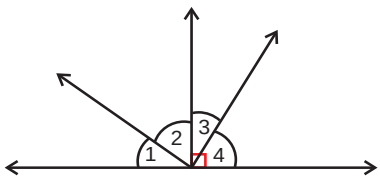


This can help you to identify complementary angles that are **adjacent**.

Don't forget:
A little square at an angle means it is a right angle, measuring 90°.

Example 3

Identify an angle in the diagram that is complementary to angle 3.



Solution

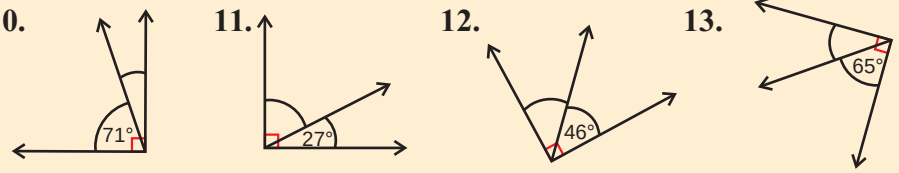
There is a **right angle** identified on the diagram, so you can look for two **adjacent angles** that form the right angle.
Angles 3 and 4 share a **common ray** and together they form the **right angle** shown.

Angle 4 is the complement of angle 3.

Guided Practice

In Exercises 10–13, find the missing angle.

10. **11.** **12.** **13.**



There Are Two Ways to Draw Complementary Angles

Don't forget:
These are very similar to the methods you saw in Section 7.2.3 for drawing supplementary angles.

Example 4

Angle C measures 15°. Draw the complement of angle C.

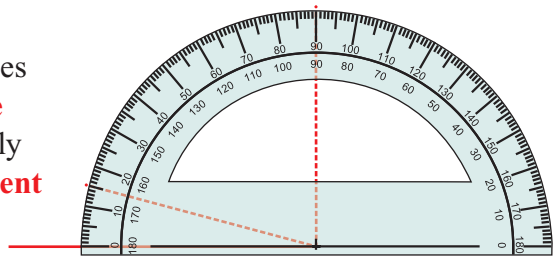
Solution

Method 1:

You can **work out** that **90 – 15 = 75**, then use a **protractor** to draw a 75° angle.

Method 2:

Use a protractor to draw angles of **15°** and **90°** with the **same vertex**. You will automatically create an **adjacent complement** to the 15° angle.



Guided Practice

In Exercises 14–19, use a protractor to draw the given angle. Then draw its complement by adding only one ray to the sketch.

14. 58° **15.** 7° **16.** 42°
17. 76° **18.** 19° **19.** 36°

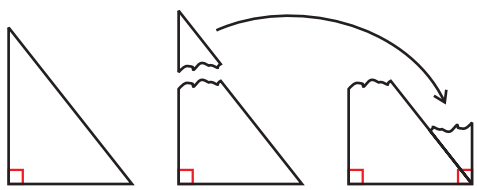
Triangles Can Contain Complementary Angles

A right triangle always has one **right angle** (90°).

Using the triangle sum, you know that the three angles in a triangle have measures that add up to **180°**. So if one of the angles has a measure of 90°, then the other two angles must add up to

$$180° - 90° = \mathbf{90°}.$$

So if one angle in a triangle is a right angle, the other two angles are **complementary angles**.



Check it out:
You could work out the size of the third angle using the triangle sum.
The third angle must be
$180° - 54° - 67° = 59°$.
But the question asks you to use complementary angles, so that's what you have to do.

Example 5

A triangle has two angles that measure 54° and 67°.
Use complementary angles to determine whether the third angle of the triangle is a right angle.

Solution

If the third angle is a right angle, the measures of the other two angles will have a sum of **90°**.

$54° + 67° =$ **121°**

So the third angle is **not** a right angle.

Guided Practice

Exercises 20–25 give the measures of two angles of a triangle.
For each pair, determine whether the third angle is a right angle.

20. 28°, 78° **21.** 65°, 35° **22.** 13°, 77°
23. 39°, 21° **24.** 41°, 49° **25.** 69°, 41°

Independent Practice

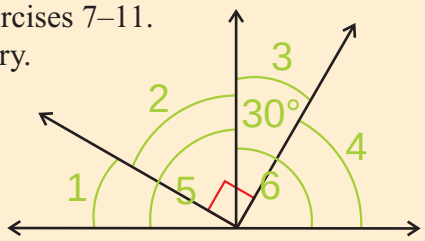
In Exercises 1–6, find the complement of the given angle.

1. 11° **2.** 26° **3.** 88°
4. 75° **5.** 52° **6.** 43°

Now try these:
Lesson 7.2.5 additional questions — p465

Use the diagram shown to answer Exercises 7–11.
Angle 3 and angle 4 are complementary.
Find the measure of each angle below.

7. Angle 6 **8.** Angle 4
9. Angle 2 **10.** Angle 5
11. Angle 1



12. Two angles are complementary. The measure of one angle is two times the measure of the other. Find the measure of each angle.

13. Two angles are complementary. The measure of one angle is 50° less than the measure of the other. Find the measure of each angle.

14. Angle R is complementary to angle S. Angle S is complementary to angle T. Is angle R complementary to angle T?
Explain your answer.

Round Up

All of this stuff about complementary angles should look quite familiar. Most of it's based on what you've already learned in this Section. If there's anything in this Lesson that you've seen before but have forgotten about, go back and take another look at the Lesson it came from to remind yourself.

Section 7.3 introduction — an exploration into:

Sorting Shapes

In this Exploration, you'll see that there are many different ways to sort shapes into groups. You'll get to use your imagination to come up with your own ways to sort them.

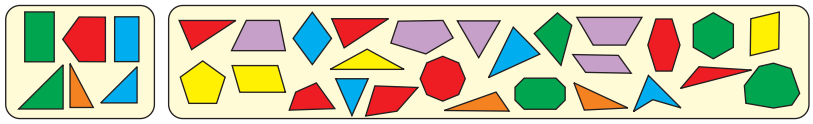
Your teacher will give you a set of **32 shapes**. You're going to **sort** them into **two groups**.

Example

Sort all 32 shapes into any two groups. What do the shapes in each group have in common?

Solution

The shapes on the **left** have a **right angle**.
The shapes on the **right** **do not have a right angle**.



Look at your shapes and put the shapes into **any two groups** — use your **imagination**. The only thing to remember is that all the shapes in a group must have **something in common**.

Exercises

1. How did you sort the shapes? What do the shapes in each group have in common?

Sort the shapes again, this time sort the shapes by the **number of sides**.

Exercises

2. What is the name for each group of shapes?

Now take the group of shapes with **three sides**, and sort these shapes into **two groups**.

Exercises

3. Describe how you sorted the three-sided polygons.

4. Draw a three-sided shape that has two equal sides and a right angle. What is the name of this shape?

Now take the group of shapes with **four sides**, and sort these shapes into **two groups**.

Exercises

5. Describe how you sorted the four-sided polygons.

6. Draw a shape that is both a rectangle and a rhombus. What is the name of this shape?



A **rhombus** is a 4-sided shape whose sides are all the same length. Like this, for example.

Round Up

There are different ways you can sort shapes — you've used a few during this Exploration. Most often in math, you'll see that polygons are sorted and named using their sides or angles.

Lesson 7.3.1

California Standard:

Measurement and Geometry 2.3
Draw quadrilaterals and **triangles from given information about them** (e.g., a quadrilateral having equal sides but no right angles, a right isosceles triangle).

What it means for you:

You'll learn one way of naming triangles and how to draw a triangle described in that way.

Key words:

- triangle
- acute angle
- obtuse angle
- right angle
- classify

Don't forget:

Right angles measure 90°.
Acute angles measure less than 90°.
Obtuse angles measure more than 90°.

Check it out:

"Classify..." means "name the type of..."

Section 7.3
Classifying Triangles by Angles

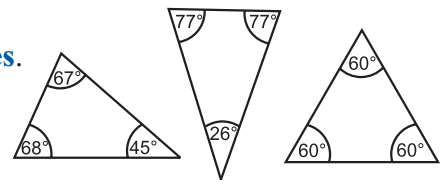
There are different types of triangles. There are a couple of ways to name the different types, depending on whether you're looking at the angles or the sides.

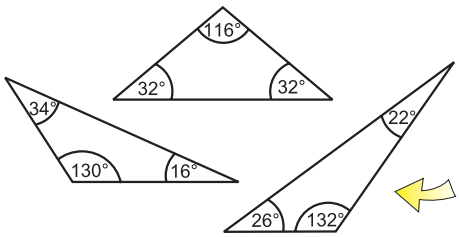
You'll learn about the two sets of names over two Lessons. In this Lesson, you'll learn how to name triangles by looking at their angles.

You Can Identify Triangles by Their Angles

Depending on the measures of their **angles**, you can sort **triangles** into **three types**:

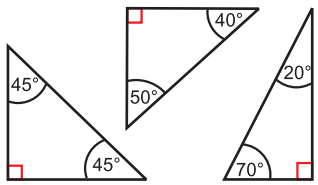
An **acute triangle** has **three acute angles**.
These triangles are acute triangles.





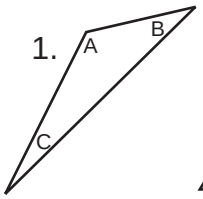
An **obtuse triangle** has an **obtuse angle**. The other two angles are acute.
These triangles are obtuse triangles.

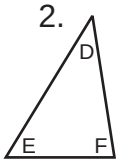
A **right triangle** has a **right angle**.
The other two angles are acute.
These triangles are right triangles.

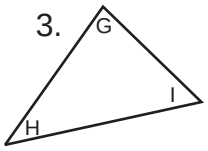


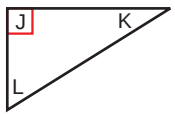
Example 1

Classify each of these triangles by its angles:

1.


2.


3.


4.


Solution

Triangle 1 is an **obtuse triangle**, because angle A is **obtuse**.

Triangle 2 is an **acute triangle**, because all three angles are **acute**.

Triangle 3 is an **acute triangle**, because all three angles are **acute**.

Triangle 4 is a **right triangle**, because angle J is a **right** angle.

Example 2

Below are the measures of the angles of three triangles. Determine whether each triangle is acute, obtuse, or right.

E. 20°, 74°, 86° F. 12°, 16°, 152° G. 90°, 40°, 50°

Solution

Triangle E is **acute**, because all three angles are **less than 90°**.

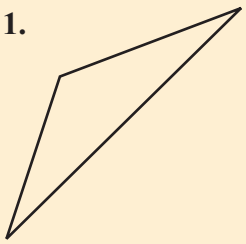
Triangle F is **obtuse**, because one angle is **more than 90°**.

Triangle G is a **right triangle**, because one angle is **90°**.

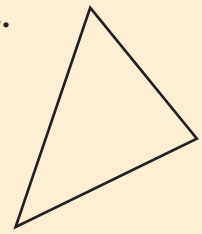
Guided Practice

In Exercises 1–6, classify each of these triangles as acute, obtuse, or right.

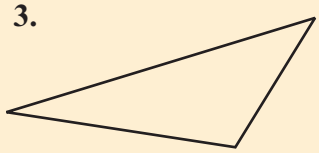
1.



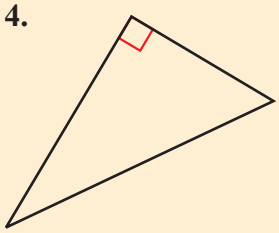
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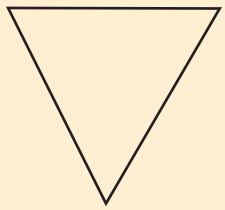
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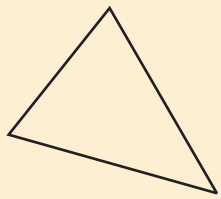
4.



5.



6.



In Exercises 7–12, the three measures of a triangle are given. Identify whether each triangle is acute, obtuse, or right.

7. 10°, 20°, 150° **8.** 18°, 90°, 72° **9.** 80°, 80°, 20°

10. 50°, 70°, 60° **11.** 34°, 60°, 86° **12.** 42°, 42°, 96°

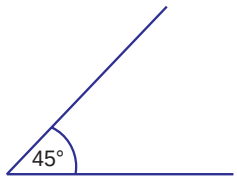
You Can Draw Triangles Using a Protractor

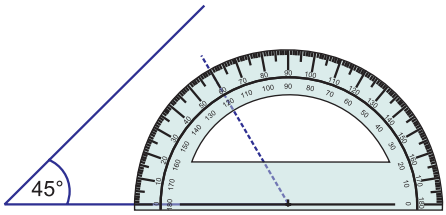
Example 3

Use a protractor to draw an acute triangle with angle measures of 60°, 45°, and 75°.

Solution

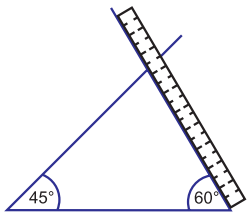
You can start by using your **protractor** to draw **any** of the three angles. You can make the sides any length you want — the question only gives you information about the angles.

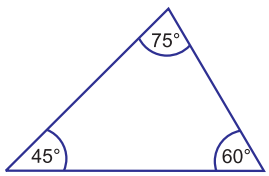




Now you need to draw the **second angle**. Use the **free end** of one of the rays you have already drawn as the vertex for the new angle.

You can now use a **straightedge** to **extend** the two rays that are not joined until they meet.
You should erase any parts of the rays that extend outside the triangle.





The third angle will **automatically** be the right measure to complete the triangle. In this example, the third angle is 75° since

$$180° - 60° - 45° = 75°.$$

Don't forget:
Look back at Lesson 7.2.1 for a reminder of how to use a protractor to draw angles.

Check it out:
It's a useful check to measure the third angle. If that is wrong, then one of the others will also be wrong.

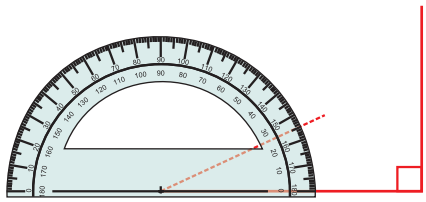
Example 4

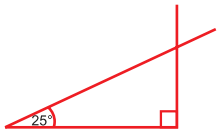
Use a protractor to draw a right triangle with a 25° angle.

Solution

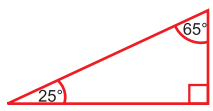
You need to draw a right triangle, so you know one of the angles is **90°**.

You can start by drawing the **right angle**. Then draw a **25°** angle at the end of one of the rays.





You can now **join up the sides** of the triangle.



Check it out:
You can use the triangle sum to work out the size of the third angle as
180° – 90° – 25° = 65°

Guided Practice

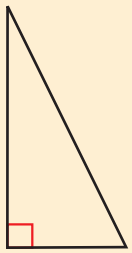
In Exercises 13–20, use a protractor to draw the triangles with the following angle measures:

13. 50°, 60°, 70°
14. 35°, 55°, 90°
15. 25°, 50°, 105°
16. 44°, 62°, 74°
17. 17°, 68°, 95°
18. 39°, 51°
19. A right triangle with one angle of 77°
20. 122°, 29°

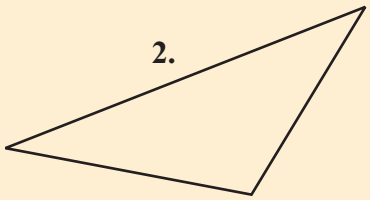
Independent Practice

In Exercises 1–8, identify each triangle shown as acute, obtuse, or right.

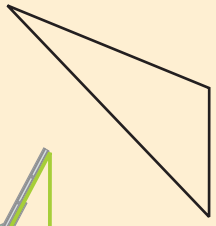
1.



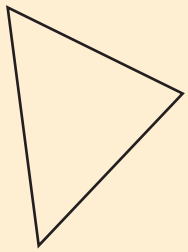
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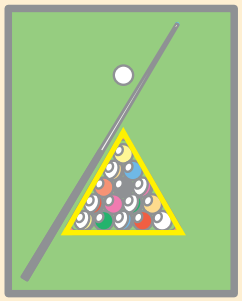
4.



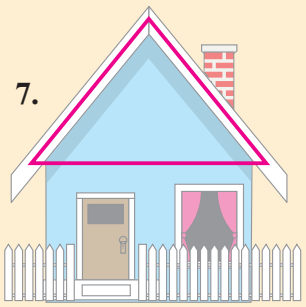
5.



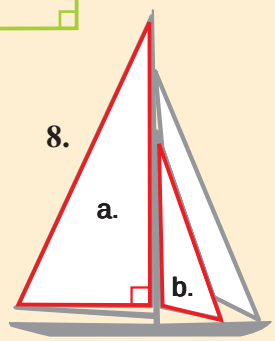
6.



7.



8.



For each statement in Exercises 9–11, determine what type(s) of triangle(s) could be being described.

9. A triangle that has exactly one right angle.
10. A triangle that has no more than two acute angles.
11. A triangle that has at least two acute angles.

In Exercises 12–13, use your protractor to draw a triangle that satisfies the condition. Then state the type of triangle you've drawn, and the measure of the missing angle.

12. Two of the three angles each measure 80°.
13. Two of the three angles each measure 25°.

Now try these:
Lesson 7.3.1 additional questions — p465

Round Up

This is just one of the ways to sort triangles into different types. The angles of a triangle only tell you part of the story. To know exactly what the triangle looks like, you also need to know about the lengths of the sides. The next Lesson will show you how side lengths are used to name triangles.

Lesson 7.3.2

Classifying Triangles by Side Lengths

California Standard:

Measurement and Geometry 2.3

Draw quadrilaterals and **triangles from given information about them (e.g.,** a quadrilateral having equal sides but no right angles, **a right isosceles triangle).**

What it means for you:

You'll learn another way of naming triangles and how to draw a triangle described in that way.

Key words:

- triangle
- equilateral
- isosceles
- scalene
- arc

Check it out:

Sides with equal numbers of tick marks are all the same length. For example, all sides with 1 tick mark are the same length, all sides with two tick marks are the same length, and so on.

Check it out:

The number of sides of equal length in a triangle is the same as the number of angles of equal measure. So all three angles in an equilateral triangle are equal, an isosceles triangle always has two angles the same, and all three angles in a scalene triangle are different.

You'll have seen the names equilateral, isosceles, and scalene triangles before in earlier grades. In this Lesson, you're going to get a reminder of what each of those names means, and you'll learn how to draw these three types of triangles.

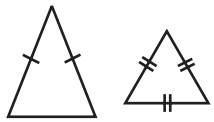
You Can Identify Triangles by Their Side Lengths

A triangle with all **three sides the same length** is called an **equilateral triangle**.

A triangle with **two sides the same length** is called an **isosceles triangle**.

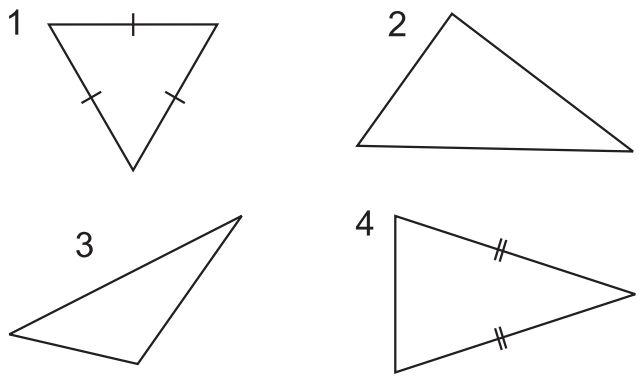
A triangle with **no sides the same length** is called a **scalene triangle**.

You can classify a triangle as equilateral, isosceles, or scalene by looking at a sketch. To make this easier, some triangles have little lines called **tick marks** on their sides.
Sides of the **same length** have **matching** tick marks.



Example 1

Classify each of these triangles by the length of its sides:



Solution

Triangle 1 is an **equilateral** triangle, because **all three** sides are the same length.

Triangle 2 is a **scalene** triangle, because **none** of the sides are the same length.

Triangle 3 is a **scalene** triangle, because **none** of the sides are the same length.

Triangle 4 is an **isosceles** triangle, because **two** sides are the same length.

You can also classify a triangle from a list of the lengths of its sides.

Example 2

Below are the measures of the side lengths of three triangles. Determine whether each triangle is equilateral, isosceles, or scalene.

A. 5 in., 8 in., 5 in. B. 9.2 cm, 9.2 cm, 9.2 cm C. 10 ft, 2 ft, 9 ft

Solution

Triangle A is **isosceles**, since **two** sides measure the same.

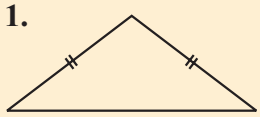
Triangle B is **equilateral**, as all **three** sides are the same length.

Triangle C is **scalene**, because the sides all have **different** measures.

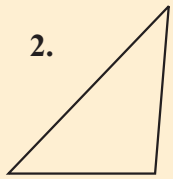
Guided Practice

In Exercises 1–6, classify each of these triangles as equilateral, isosceles, or scalene.

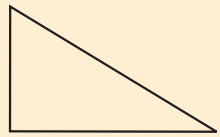
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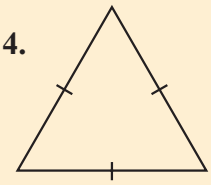
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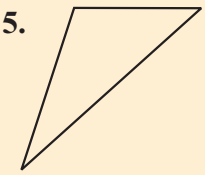
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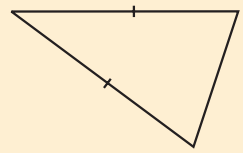
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5.



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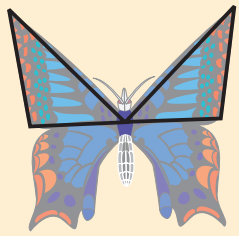


In Exercises 7–12, the lengths of the sides of triangles are given. Identify whether each triangle is equilateral, isosceles, or scalene.

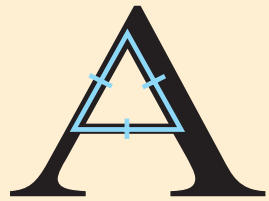
7. 6 cm, 6.5 cm, 6.1 cm
8. 3 ft, 3 ft, 4 ft
9. 10 in., 20 in., 28 in.
10. 5 yd, 5 yd, 5 yd
11. 0.4 m, 0.4 m, 0.4 m
12. 8 in., 3 in., 6 in.

In Exercises 13–16, identify whether the highlighted triangles are equilateral, isosceles, or scalene.

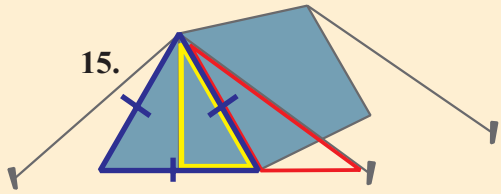
13.



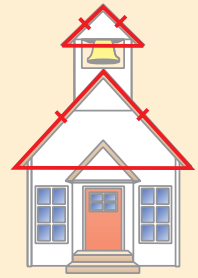
14.



15.



16.



You Can Draw Triangles Using a Ruler and Compass

In the last Lesson, you used a **protractor** to draw triangles with certain **angle** properties. To draw triangles based on the measures of their **side lengths**, you'll need different tools.

Check it out:
If you use a compass to draw a curve that is not a full circle, the curve is called an arc.

Example 3

Use a compass and a ruler to draw a scalene triangle with side lengths 7 cm, 3 cm, and 9 cm.

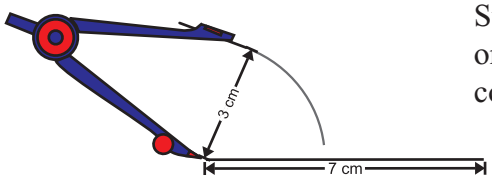
Solution

You could try to draw the triangle using just a **ruler**.
After some **trial and error**, you could probably get it right.

By using a **compass** and a ruler together, you can get the drawing right the first time.

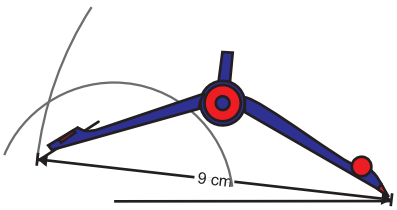
Step 1: Use your **ruler** to measure and draw a line **7 cm** long.

Step 2: Use the ruler to measure **3 cm** between the **compass tip** and the **pencil tip**.



Step 3: Place the **compass tip** on one **end** of the 7 cm line. Use the compass to **make an arc**.

Step 4: Open the compass to **9 cm**. Place the compass tip on the **opposite end** of the 7 cm line. Make another arc. **Continue** the arc until it **crosses the first arc** you made.



Step 5: Use the **straight edge** of your ruler to **connect the end points** of the 7 cm line to the point where the arcs cross.



You now have a **scalene triangle** with side lengths **7 cm, 3 cm, and 9 cm**.

Guided Practice

In Exercises 17–22, draw a triangle with the side lengths described.

17. 4 cm, 5 cm, 6 cm
18. 3 in., 3 in., 3 in.
19. 5 cm, 7 cm, 8 cm
20. 5 cm, 5 cm, 8 cm
21. 2 in., 5 in., 5 in.
22. 4 in., 7 in., 5 in.

There is a Method for Drawing Each Type of Triangle

If you are not given **specific side lengths**, you can still draw an **example** of **each type** of triangle. There are slightly different methods for drawing equilateral, isosceles, or scalene triangles.

Example 4

Use a compass and ruler to draw an **equilateral** triangle.

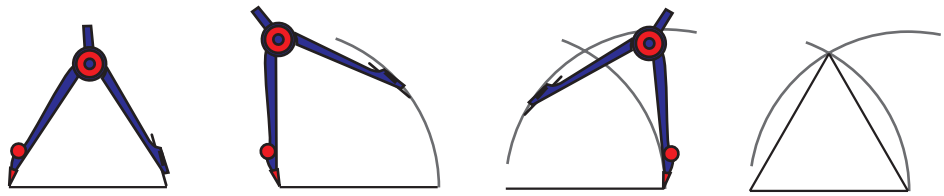
Solution

Step 1: Draw a **straight line**. Open your compass and place the **compass tip** on **one end** of the line and the **pencil tip** on the **other end**.

Step 2: With the compass tip still on the end of the line, draw an **arc**.

Step 3: Keep the compass opening **the same** and place the compass tip on the **other end** of the line. Draw another arc. Make the arcs long enough so that they **cross**.

Step 4: Use the straight edge of your ruler to **connect the ends** of the line to the point where the arcs **cross**.



Step 1 Step 2 Step 3 Step 4

Example 5

Use a compass and ruler to draw an **isosceles** triangle.

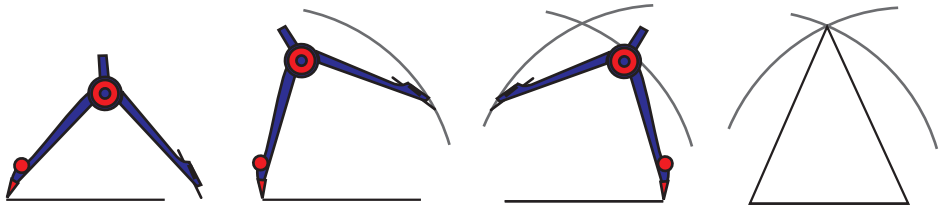
Solution

Step 1: Draw a line. Open your compass to **any distance more than half** the length of the line, but **not the same** length as it.

Step 2: Place the compass tip on **one end** of the line and draw an **arc**.

Step 3: Keep the compass opening **the same** and repeat Step 2 from the **other end** of the line. Make the arcs long enough so that they **cross**.

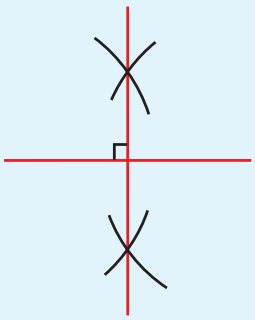
Step 4: Use the straight edge of your ruler to **connect the ends** of the straight line to the point where the arcs **cross**.



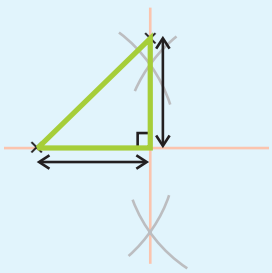
Step 1 Step 2 Step 3 Step 4

Check it out:

You can draw a right isosceles triangle using a method similar to that shown in Example 5. The first step is to draw a right angle.
1) Draw a line.
2) Put the compass tip on one end of the line, and draw two arcs — one either side of the line.
3) Put the compass tip on the other end of the line, and again, draw two arcs — one either side of the line.



4) Join the crossing points of the arcs to form a right angle.
5) Now measure points on the two lines, equal lengths away from your right angle. Joining these gives you a right isosceles triangle.



Example 6

Use a compass and ruler to draw a **scalene** triangle.

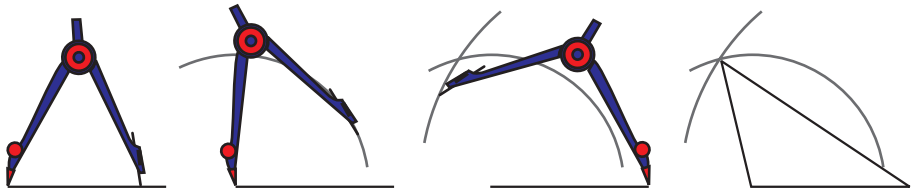
Solution

Step 1: Draw a line. Open your compass to any distance that is **not equal to the length of the original line**.

Step 2: Place the compass tip on **one end** of the line and draw an **arc**.

Step 3: Place the compass tip on the **other end** of the line. **Change** the compass opening and draw a **new arc**. The new compass opening can be **any length** other than the length of the original line, so long as the second arc **crosses the first one**.

Step 4. Use the straight edge of your ruler to **connect the ends** of the line to the point where the arcs **cross**.



Step 1 Step 2 Step 3 Step 4

Guided Practice

23. Use a compass and a ruler to draw an equilateral, an isosceles, and a scalene triangle, each with a base measuring 4 cm.

24. Leila was using her compass to construct an isosceles triangle, but she was not able to make a triangle. Her work is shown here. Explain what happened and what she should do instead.



Independent Practice

Now try these:
Lesson 7.3.2 additional questions — p465

In Exercises 1–4, draw a triangle with the side lengths described. Determine whether each triangle is equilateral, isosceles, or scalene.

1. 6 cm, 6 cm, 6 cm
2. 8 in., 5 in., 6 in.
3. 3 in., 4 in., 5 in.
4. 5 cm, 5 cm, 7 cm

5. Use your compass to construct several equilateral triangles. Measure each angle of your triangles. Look for a pattern in the measures of the angles.

6. Is it possible for a triangle to be an obtuse equilateral triangle? Explain using your answer to Exercise 5.

Round Up

This way of naming triangles is quite common, so you should get to know the names equilateral, isosceles, and scalene really well. Practice the methods for drawing the three types of triangles.

Lesson 7.3.3

Types of Quadrilaterals

California Standard:

Measurement and Geometry 2.3
Draw quadrilaterals and triangles **from given information about them** (e.g., a quadrilateral having equal sides but no right angles, a right isosceles triangle).

What it means for you:
You'll learn the names and special properties of some four-sided shapes, also known as quadrilaterals.

Key words:

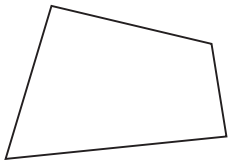
- quadrilateral
- rectangle
- square
- parallelogram
- rhombus
- trapezoid
- isosceles

Don't forget:
Matching tick marks show sides of the same length.

In the last couple of Lessons, you've been looking at triangles. Now you're going to move on to four-sided shapes. These are also called quadrilaterals.

A Quadrilateral Has Four Sides

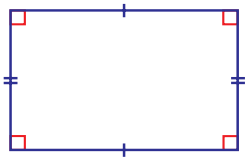
Polygons are named by the number of angles or number of sides they have. You know that **3-sided** figures with **3 angles** are called **triangles**.

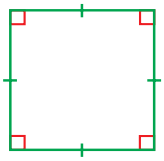


When a polygon has **4 sides** and **4 angles**, it is called a **quadrilateral**.

Rectangles are a type of **quadrilateral**.

In a **rectangle**, the **opposite** sides are the **same length**. All the **angles** in a rectangle measure **90°**.





A **square** is a special type of rectangle where **all the sides** are the same length. So squares are **quadrilaterals** too.

A Quadrilateral's Angle-Measures Sum to 360°

The **angles** inside a rectangle have measures that add up to $4 \times 90° = \mathbf{360°}$.

In fact, the angles inside **any quadrilateral** add up to **360°**.

Example 1

Find the measure of the missing angle in this quadrilateral.



Solution

You know the measures of three of the angles.
You also know that the **sum** of the four angles' measures is **360°**.
So you can write an **equation** to find the missing angle.

$50° + 60° + 100° + n = 360°$ — **Let n = measure of missing angle**

$210° + n = 360°$

$n = 360° - 210°$ — **Rearrange the equation**

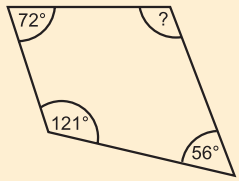
$n = \mathbf{150°}$ — **Solve to find n**

The missing angle measures **150°**.

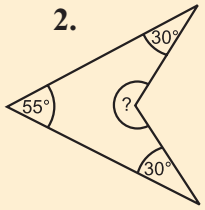
Guided Practice

In Exercises 1–4, find the missing angle measure in each quadrilateral.

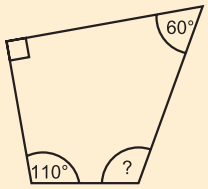
1.



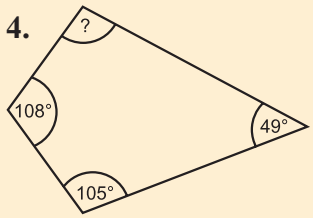
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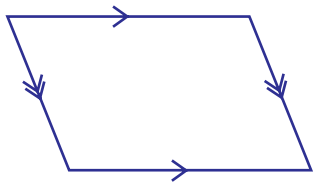


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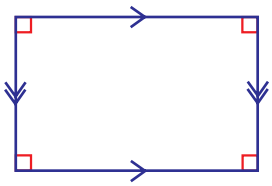


A Parallelogram Has Two Pairs of Parallel Sides

A **parallelogram** is any quadrilateral with **two pairs of parallel sides**. Parallel sides are often marked with **matching arrows**.



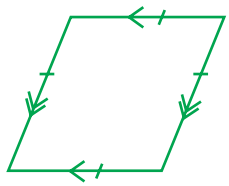
Don't forget:
Parallel lines are lines that always stay the same distance apart.



Opposite sides of a rectangle are **parallel**. So a rectangle is a special type of parallelogram.

A parallelogram with all **four** sides the **same length** is called a **rhombus**.

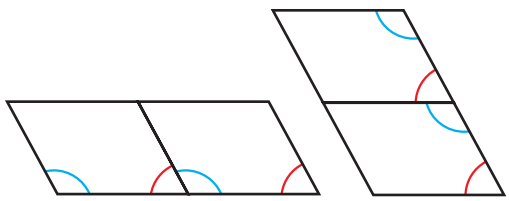
So a square is a special type of rhombus.



Check it out:
The plural of rhombus is rhombi.

In a parallelogram or rhombus, the measures of the angles at **either end** of **any side** add up to **180°**.

You can see this if you place two identical parallelograms next to each other.



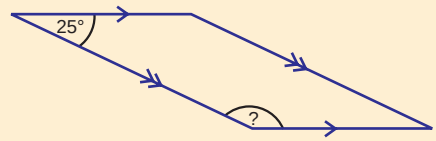
Don't forget:
For a reminder about linear pairs of angles, see Lesson 7.2.2.

Where two angles meet, they make a **linear pair** of angles, so their measures add up to **180°**.

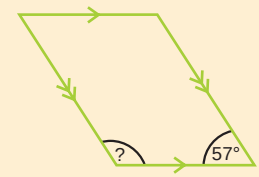
Guided Practice

In Exercises 5–8, find the measure of the missing angle.

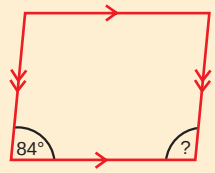
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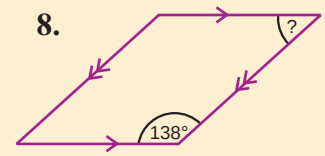
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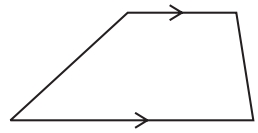


8.



A Trapezoid Has Exactly One Pair of Parallel Sides

A **trapezoid** is a quadrilateral with **exactly one** pair of **parallel sides**. The figure below is a trapezoid.

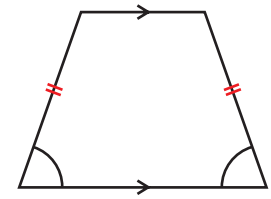


The **parallel sides** of a trapezoid are called the **bases**.

The bases of a trapezoid are **never** the same length.

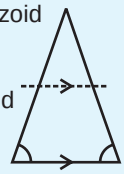
Check it out:
The bases of a trapezoid must be different lengths. If they were the same length, you'd have a parallelogram.

If the **nonparallel sides** of a trapezoid are the **same length**, then the trapezoid is called an **isosceles trapezoid**.



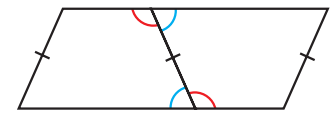
The **angles** at the ends of a **base** of an isosceles trapezoid have **the same measure**.

Check it out:
An isosceles trapezoid is like an isosceles triangle that's had the top bit "chopped off" parallel to the base.



The measures of the angles at the ends of a **nonparallel side** of an isosceles trapezoid add up to **180°**.

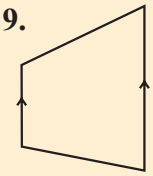
You can see this by putting two isosceles trapezoids together. When the angles at the ends of the nonparallel sides are **lined up** next to each other, they make a **linear pair**.



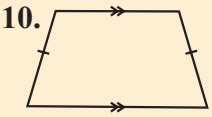
Guided Practice

In Exercises 9–14, say which of these shapes are trapezoids. If they are not, give a reason.

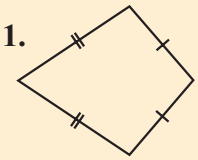
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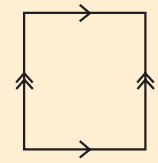
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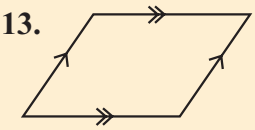
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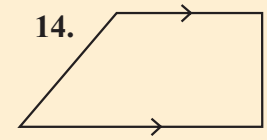
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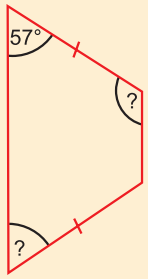
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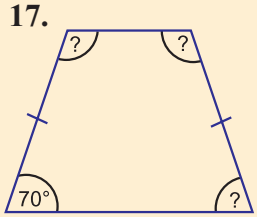
15. Which one of the trapezoids above is an isosceles trapezoid?

In Exercises 16–18, find the missing angle measures in each trapezoid.

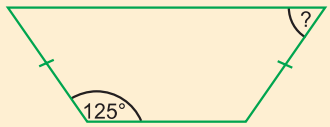
16.



17.



18.



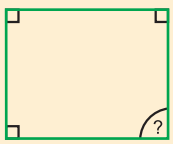
Independent Practice

In Exercises 1–6, complete the sentences by filling in the blank with the correct word.

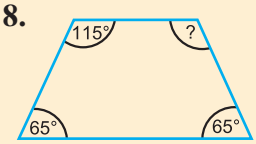
1. A rectangle is a parallelogram with four _____ angles.
2. A rhombus with four right angles is called a _____.
3. A rectangle with four equal-length sides is called a _____.
4. A quadrilateral is a _____ -sided polygon.
5. Isosceles trapezoids have at least _____ sides of equal length.
6. A trapezoid has exactly ____ pair of sides that are parallel.

In Exercises 7–9, find the measure of the missing angle.

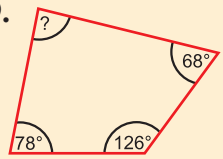
7.



8.



9.



In Exercises 10–14, say whether the statements are true or false. If false, explain why.

10. A trapezoid is a quadrilateral.
11. A polygon can be both a trapezoid and a parallelogram.
12. In an isosceles trapezoid, the parallel sides can be the same length.
13. A parallelogram cannot have any right angles.
14. A quadrilateral must be either a trapezoid or a parallelogram.

15. If all the angles in a quadrilateral are equal, how much does each angle measure?

16. One side of a parallelogram measures 16 cm.
What do you know about the measure of the opposite side?

17. Use the clues below to determine the polygon being described.
I have at least 1 pair of parallel sides.
I do not have any right angles.
My sides are all the same length.
What am I?

Now try these:
Lesson 7.3.3 additional questions — p466

Round Up

The names of the special types of quadrilaterals in this Lesson should be familiar from lower grades. Try to remember the special properties of each type of quadrilateral — they will be useful when you learn how to draw them in the next Lesson.

Lesson 7.3.4

Drawing Quadrilaterals

California Standard:

Measurement and Geometry 2.3

Draw quadrilaterals and triangles **from given information about them (e.g., a quadrilateral having equal sides but no right angles**, a right isosceles triangle).

What it means for you:

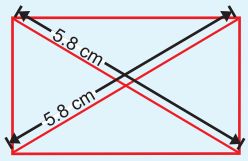
You'll find out how to draw the special quadrilaterals you learned about in the last Lesson.

Key words:

- quadrilateral
- rectangle
- square
- parallelogram
- rhombus
- trapezoid
- isosceles

Check it out:

The diagonals from one corner of a rectangle to the other are always the same length. So you can check that you have drawn a true rectangle by measuring the diagonals to make sure they are both the same.



In the last Lesson, you met some of the most important of the special types of quadrilaterals. In this Lesson, you'll learn how to draw them.

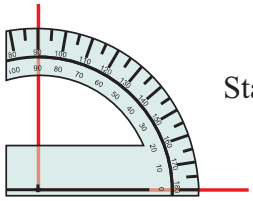
Use a Protractor and Ruler to Draw Rectangles

You can draw **rectangles** and **squares** using a protractor and ruler.

Example 1

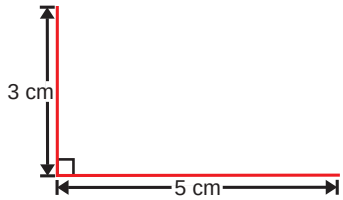
Draw a rectangle measuring 3 cm by 5 cm.

Solution



Start by using a **protractor** to make a **right angle**.

Then use a ruler to **extend** the rays to **3 cm** and **5 cm**.





Using your protractor again, make new right angles at the **free ends** of the sides you have drawn.

Now **extend** the new rays until they **join up** and complete the rectangle.



Guided Practice

In Exercises 1–6, draw a rectangle with the measurements described.

1. 2 in. by 5 in.
2. 3 cm by 4 cm
3. 1 cm by 5 cm
4. 4 in. by 4 in.
5. 6 in. by 1 in.
6. 3.5 cm by 8.5 cm

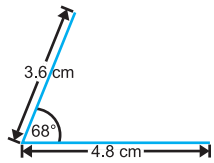
Use a Protractor and Ruler to Draw Parallelograms

Example 2

Draw a parallelogram with side lengths 3.6 cm and 4.8 cm.

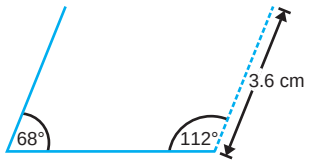
Solution

First, **draw one side** of the parallelogram. Then draw the second side starting at **one end** of the first. The angle between the sides can be any size you want.



Use a protractor to **measure the angle** between the sides.

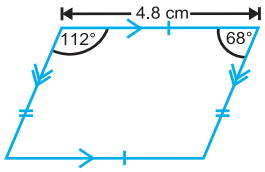
Now make an angle that is **supplementary** to the one you've measured. In this case, that will measure 180° – 68° = **112°**.



Draw the supplementary angle at the **free end** of **either** of the sides you have already drawn.

Then **extend** the new ray so that it is the same length as the side it is **opposite** to.

Finally, **join up** the two remaining free ends. The final side will **automatically** be parallel to the side it is opposite, and the same length.



Don't forget:

Supplementary angles add up to 180°. See Lesson 7.2.3 for more.

Check it out:

You can use exactly the same method to draw a rhombus (a quadrilateral with 4 sides of the same length).



Check it out:

If you are given the angle measures instead of side lengths, you'll need to start by using a protractor to draw an angle. Then extend the sides to whatever length you like before drawing the next angle.

Guided Practice

In Exercises 7–10, draw a parallelogram with the given measurements.

7. Side lengths 3 in. and 5 in.

8. Side lengths 5.0 cm and 6.0 cm.

9. Angles measuring 77° and 103°.

10. One side of length 4 in. and at least one angle of 65°.

There Are Two Ways to Draw Isosceles Trapezoids

Example 3

Use a protractor and ruler to draw an isosceles trapezoid.

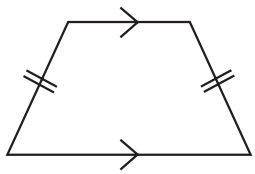
Solution

First **draw a line** for one of the **bases**.

Use your protractor to create **two angles** of **equal measure** from the endpoints of the base.

Make the sides the **same length**, then **connect** the fourth side. This way the bases will be **parallel**.





Don't forget:
For a reminder of how to construct an isosceles triangle, see Lesson 7.3.2.

Don't forget:
You could use the compass to mark points the same distance from the corners.

Example 4

Use a compass and straight edge to draw an isosceles trapezoid.

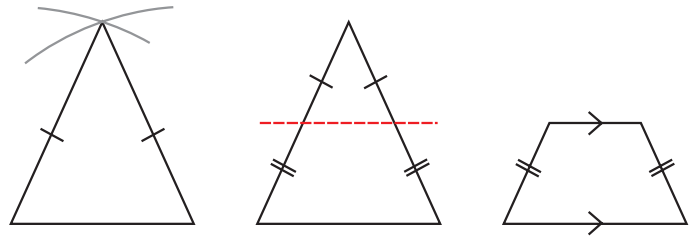
Solution

Begin by constructing an **isosceles triangle**.

Measure and mark a point on each of the two **equal length sides**.
Both points should be the **same distance** from the **base** of the triangle.

Join the points to make a line **parallel** to the base.

Erase the corner opposite the base of the triangle.
The remaining figure is an **isosceles trapezoid**.



Guided Practice

In Exercises 11–14, draw an isosceles trapezoid that has the given measurements.

11. One base of length 4 cm.
12. Two angles measuring 50°.
13. One base of length 3 in. and two sides of length 2 in.
14. One base of length 4 in. and two angles measuring 102°.

Now try these:
Lesson 7.3.4 additional questions — p466

Independent Practice

In Exercises 1–8, draw the shapes described.

1. A square with side length 1 in.
2. A rectangle measuring 6.8 cm by 3.9 cm.
3. A parallelogram in which one angle measures 140°.
4. A rhombus in which one of the angles measures 130°.
5. A rhombus with side length 4.4 cm.
6. A parallelogram with one angle measure of 56° and one side length of 5 in.
7. An isosceles trapezoid with two angles measuring 72° and two sides of length 2 in.
8. An isosceles trapezoid with one base of length 4.3 cm and two sides of length 2.6 cm.

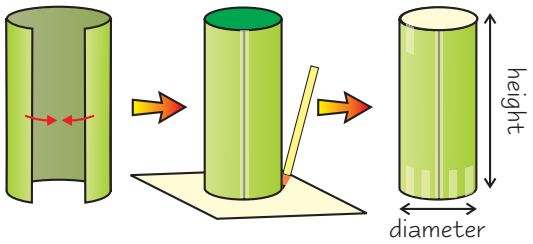
Round Up

The best way to get the hang of drawing these shapes is to practice doing it. It is important that you understand flat shapes before you move on to 3-D shapes in the next Section of this Chapter.

Section 7.4 introduction — an exploration into:

Building Cylinders

This Exploration involves some light engineering — you're going to build a cylinder. But it can't be just any cylinder — it has to be a cylinder with a particular volume. And if that's too easy, you've then got to try and do it using the minimum amount of materials. Just like real engineering.



You are going to make some **cylinders** out of **paper**.

- Roll up a **rectangular** piece of paper, and tape the edges to form the **walls**.
- Make the **bases** by carefully **drawing round** your paper roll onto a sheet of paper. Then **cut out** these circles, and **attach** them to the cylinder.

Your **aim** is to build a cylinder with a **volume between 460 cm³ and 480 cm³**.

Example

For each cylinder you build, record the relationship between the diameter, height, and volume.

Solution

The **height** and **diameter** are marked on the diagram above. Use a table to **record** these for each cylinder you make.

To find the **volume**, fill your cylinder with **rice**. Then empty the rice into a **measuring jug** and record the volume of rice.

diameter (cm)	height (cm)	volume (cm³)
5	15	295
10	15	1178
10	5	393

Exercises

1. Calculate the base area of each of your cylinders.
Describe the relationship between the base area, height, and volume.

Use the formulas "$r = \frac{d}{2}$" and "$A = \pi r^2$."

Now try to make a cylinder with a **volume of 460-480 cm³**, but using **as little paper as possible**. **Experiment**, by making some different-sized cylinders. Then find their **volume** and **surface area**. Add an **extra column** to your table to **record** all your results.

Example

Find the surface area of a cylinder with diameter 8 cm and height 9 cm.

Solution

The surface area is the total area of the **one rectangular** piece of paper, plus the **two circular pieces**. Work them out like this:

So the total surface area is 226 + 50.3 + 50.3 = **326.6 cm²**

radius (r) = 4 cm → Area $= \pi r^2$ = 50.3 cm²

9 cm

Area of rectangular piece
= cylinder circumference × height
= ($\pi \times d$) × h
= (3.14 × 8) × 9 = 226.0 cm²

Exercises

2. What do you think is the least possible surface area of a cylinder with volume 460-480 cm³?

Round Up

That was a very difficult task, but it's a good review of circles. And it did get you working with 3D shapes. That's good, because there are many more 3D shapes in this Section of the book.

Lesson 7.4.1

Section 7.4
Three-Dimensional Figures

The 3D shapes you're going to see in this Lesson should be familiar. You'll have seen them before in math lessons in earlier grades, and also all around you in real life.

California Standard:

Measurement and Geometry 1.3
Know and use the formulas for the volume of **triangular prisms and cylinders** (area of base × height); compare these formulas and **explain the similarity between them and** the formula for the volume of **a rectangular solid**.

What it means for you:
You'll see some examples of 3D figures, with their names. You'll learn about faces, edges, and vertices of 3D figures.

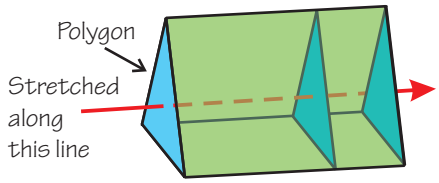
Key words:
- rectangular prism
- cube
- triangular prism
- cylinder
- face
- edge
- vertex
- polygon

Don't forget:
A polygon is a flat shape with straight sides. Polygons include triangles, rectangles, parallelograms, and trapezoids — plus many other shapes.

A Prism Is a Special Type of 3D Figure

A **prism** is a type of 3D shape that looks like a **polygon** that's been **stretched out** along a straight line. A prism is exactly **the same size and shape** all the way through.

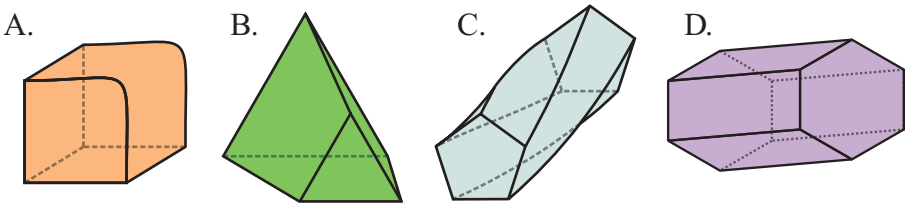
If you cut through a prism in line with one end, the slice is always the **same shape and size** and the **same way around**.



Example 1

Which of these figures is a prism? Explain why the others are not.

A. B. C. D.



Solution

The ends of Figure A have a **curved side**, so they are **not polygons**. So Figure A **isn't** a prism.

Figure B **isn't** a prism because the triangles at the front and back of the figure are **different sizes**. The figure will be the same shape all the way through, but **not** the same size.

Figure C **isn't** a prism. If you cut through it as shown here, the slice is the **same shape** as the end, but **not** the same way up, so it's not exactly the same.

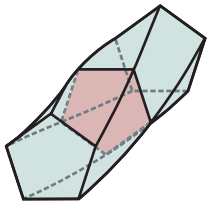
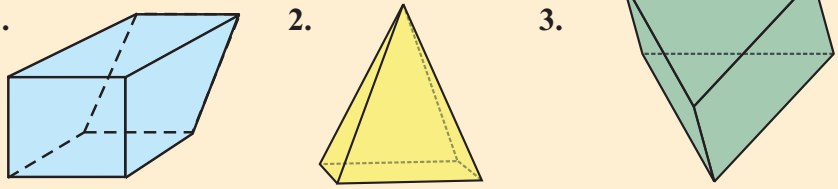


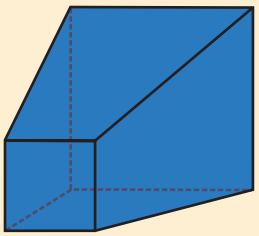
Figure D **is** a prism. It's the same shape, the same size, and the same way around **all the way through**.

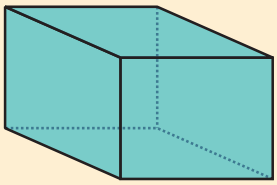
Guided Practice

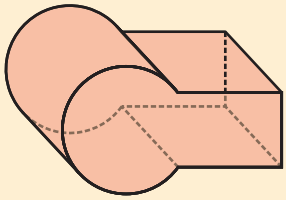
In Exercises 1–6, identify whether or not the shapes are prisms. If they are not, give reasons why.

1. 2. 3.



4. 

5. 

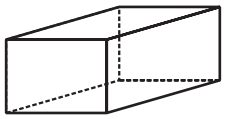
6. 

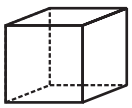
Some Prisms Have Special Names

The sides of a 3D figure are called faces.
The polygons at the ends of a prism are called **bases**.

Check it out:
Another name for a rectangular prism is a cuboid.

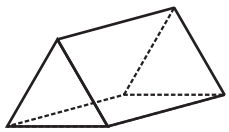
A 3D figure in which **all** the faces are **rectangles** is called a **rectangular prism**.

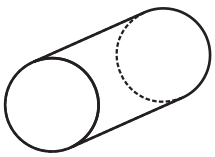




A **cube** is a special type of rectangular prism where all the faces are **squares**.

A **triangular prism** is a prism in which the bases are **triangles**. The other faces in the prism are **rectangles**.

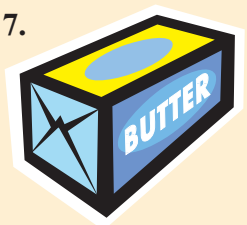




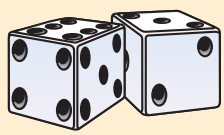
This shape isn't actually a prism, but it's very similar. A **cylinder** is a three-dimensional figure with **two circular bases** that are parallel. But a cylinder is not actually a prism because circles are not polygons.

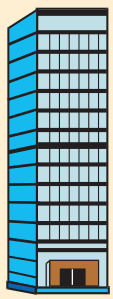
Guided Practice

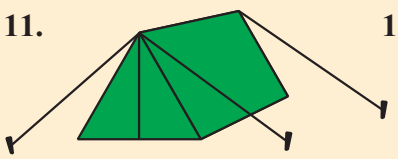
In Exercises 7–12, name the 3D figures shown.

7. 

8. 

9. 

10. 

11. 

12. 

3D Figures Have Faces, Edges, and Vertices

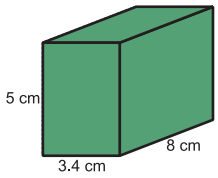
The **line** where two faces of a figure meet is called an **edge**.
A **point** where edges meet is called a **vertex**.

Check it out:
A vertex is the mathematical name for what in everyday language we just call a corner point.

Check it out:
The plural of vertex is vertices.

Example 2

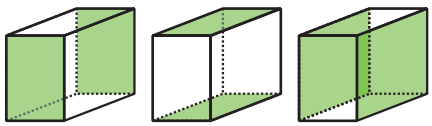
Name this figure and determine how many faces, edges, and vertices it has.



Solution

This is a **rectangular prism**, since all the faces are **rectangles**.

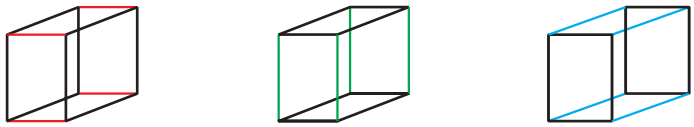
When you count the faces, edges, and vertices, you need to remember to count the ones you can't see on the picture, but you know are there.



In the picture you can see **3 faces**. Each one is opposite a **matching face** that you can't see.

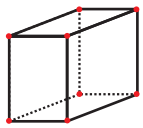
So there are **6 faces** in total.

There are **4 edges** of length 3.4 cm, **4** of length 5 cm, and **4** of 8 cm.



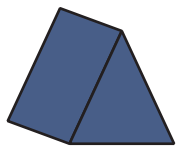
That makes a total of **12 edges**.

Finally, the prism has **8 vertices**. You can see **7** of them on the original picture. The eighth is on the **other side**.



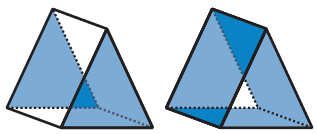
Example 3

Name this figure and determine how many faces, edges, and vertices it has.



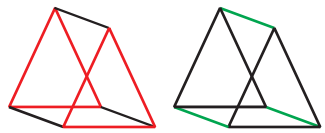
Solution

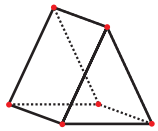
This is a **triangular prism**. There are 2 **parallel triangles** and all the other faces are **rectangles**.



It has **5 faces**. It has **2** faces that are **triangles**, and **3** that are **rectangles** joining the bases together.

Each **triangle** has **3 edges**. There are another **3 edges** where the rectangular faces meet.
So there are **9 edges** in total.

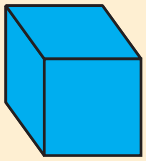


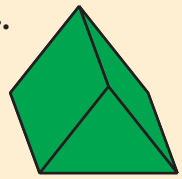


There are **6 vertices**. You can count the **3 vertices** on the front triangle and **3** on the back triangle.

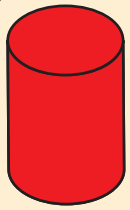
Guided Practice

In Exercises 13–16, say how many vertices each figure has.

13. 

14. 

15. 

16. 

17. Charlie is making a cube-shaped box from square panels of wood. How many square panels does she need?

Independent Practice

In Exercises 1–3, say whether each statement is true or false. If any are false, explain why.

1. The faces of a triangular prism are triangles.

2. A cube is not a rectangular prism.

3. A cylinder is not a prism.

In Exercises 4–8, use the word face(s) or base(s). Choose the best word to complete each sentence.

4. Prisms are named by their _____. All the other _____ are rectangles.

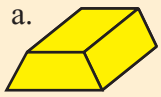
5. In a cylinder, the _____ are circles.

6. In a triangular prism, there are three rectangular _____.

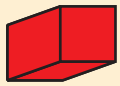
7. A _____ is one of the _____.

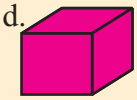
8. In rectangular prisms, all the _____ are rectangles.

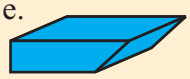
9. Identify which of the figures are rectangular prisms.

a. 

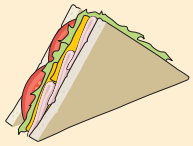
b. 

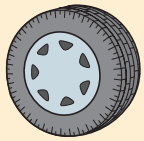
c. 

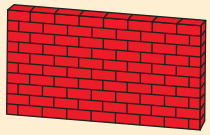
d. 

e. 

In Exercises 10–15, name the 3D figures shown.

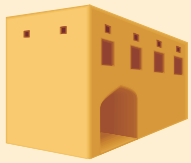
10. 

11. 

12. 

13. 

14. 

15. 

Now try these:
Lesson 7.4.1 additional questions — p466

Round Up

You will have seen these special types of 3D figures before in earlier grades. You should make sure you understand all the facts you have seen in this Lesson. They'll be useful in the next few Lessons as you learn about the volume of 3D figures.

Lesson 7.4.2

Volumes of Rectangular Prisms

California Standard:

Measurement and Geometry 1.3
Know and use the formulas for the volume of triangular prisms and cylinders (area of base × height); compare these formulas and explain the similarity between them and **the formula for the volume of a rectangular solid**.

What it means for you:
You'll learn a formula that you can use to work out how much space there is inside a rectangular prism.

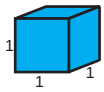
Key words:
- volume
- rectangular prism
- cube
- base
- area
- height

It's really useful to be able to work out the volume of different solid figures. The easiest type of figure to work out the volume of is a rectangular prism. That's what you'll learn about in this Lesson.

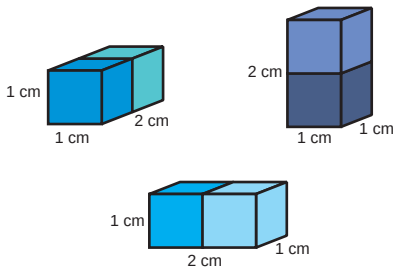
Volume Is the Amount of Space Inside a Figure

The amount of **space inside** a 3D figure is called the **volume** of the figure.

Volume is measured in **cubic units**. One cubic unit is the volume of a **unit cube** — a cube with a side length of **1 unit**.



When you work out the volume of a 3D figure, you are working out how many unit cubes would fit inside it.



If you put two 1 cm cubes together, you make a **rectangular prism** with side lengths 1 cm, 1 cm, and 2 cm.

The **volume** of the prism is the total of the two cubes together, **2 cm^3**.

Check it out:

A unit cube could have side lengths of 1 cm, 1 m, 1 yd... it all depends on the units you're using. For bigger volumes, you need bigger units — you might measure volumes using a 1-meter unit cube.

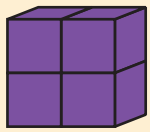
Guided Practice

The rectangular prisms shown in Exercises 1–6 are made up of 1 cm cubes. Find the volume of each prism.

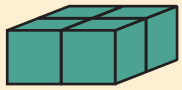
1.



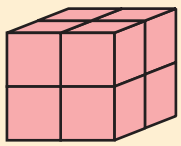
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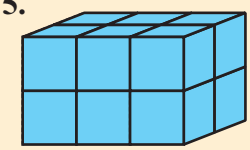
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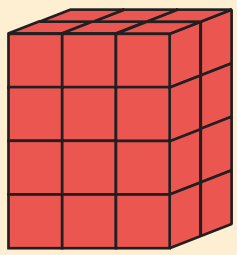
4.



5.



6.



In Exercises 7-10, choose a reasonable estimate of each volume from the three choices.

7. Volume of a bathroom	13 cm^3	9 m^3	90 m^3
8. Volume of a bar of soap	8 in^3	80 in^3	0.8 in^3
9. Volume of a bathtub	0.75 yd^3	7.5 yd^3	75 yd^3
10. Volume of a medicine cabinet	10 in^3	900 in^3	1 in^3

You Can Calculate Volume Using a Formula

For larger prisms, it isn't possible to find the volume by counting cubes. It is more convenient to use a **formula** to **work out** the volume.

$V = Bh$

This is the formula you need to find the volume of a rectangular prism.

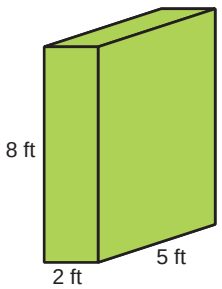
B represents the **area** of one of the prism's **bases**.

h stands for the **height** of the prism.

In a rectangular prism, you can pick **any pair** of parallel faces as the **bases**. The height is the **distance** between the bases.

Example 1

Find the volume of this rectangular prism.



Solution

You need to start by choosing a rectangle as the base.

For example, you might choose the **2 ft by 5 ft** rectangle as the **base**, making the **height 8 ft**.

You can now find the volume using the formula $V = Bh$

$V = Bh$

$V = (l \times w) \times h$ — Replace B with rectangle area formula

$V = (2 \times 5) \times h$ — Substitute in length and width of base

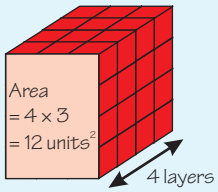
$V = 10 \times h$

$V = 10 \times 8$ — Substitute in height of the prism

$V = 80$

So the volume of the prism is 80 cu. ft, or **80 ft³**.

Check it out:



Think of it like this...

The area of the base (B) is the number of unit cubes in one layer (here, 4 × 3 = 12).

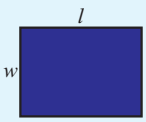
The volume V (the total number of unit cubes) is the number of cubes in one layer multiplied by the number of layers.

Don't forget:

The area of a rectangle is given by the formula

$A = lw$

where l and w are the length and width of the rectangle.



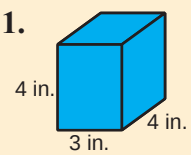
Check it out:

1 ft³ or 1 cu. ft is 1 cubic foot — the volume of a unit cube whose sides are 1 foot long.

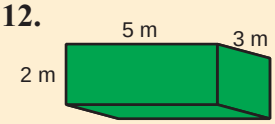
Guided Practice

In Exercises 11–16, use the formula $V = Bh$ to find the volume of each prism.

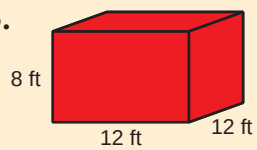
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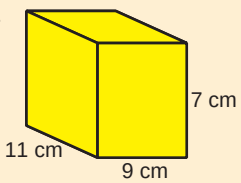
12.



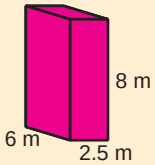
13.



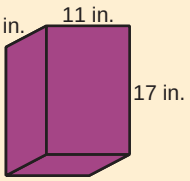
14.



15.



16.



You Can Find the Volume Without a Picture

You don't have to have a drawing of a prism to find its volume.

You can use the formula $V = Bh$ to find the volume of a rectangular prism when only the lengths of the sides are given.

Example 2

Find the volume of a rectangular prism with length of 3.4 cm, width of 5.6 cm, and height of 3 cm.

Solution

Use the formula $V = Bh$.

$V = (l \times w) \times h$ — Replace B with rectangle area formula

$V = (3.4 \times 5.6) \times 3$ — Substitute values of l, w, and h

$V = 19.04 \times 3$

$V = 57.12$

The volume of the prism is $57.12\ \text{cm}^3$.

Check it out:

The formula for the volume of a rectangular prism is often written

$V = l \times w \times h$

(or $V = lwh$)

— where l, w, and h are the length, width, and height.

Guided Practice

Exercises 17–22 each give the length, width, and height of a rectangular prism. Calculate the volume of each prism.

17. 4 cm, 9 cm, 1 cm

18. 3 ft, 7 ft, 10 ft

19. 8 in., 8 in., 11 in.

20. 5.5 cm, 7 cm, 9.5 cm

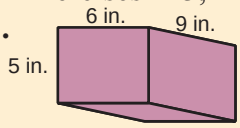
21. 1.2 m, 5.2 m, 6.9 m

22. 3.25 in., 4.5 in., 6 in.

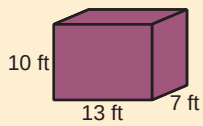
Independent Practice

In Exercises 1–3, find the volume of each rectangular prism.

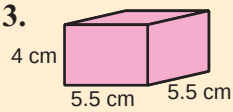
1.



2.



3.



4. The area of the base in a rectangular prism is 35 cm^2 and the height is 2 cm. What is the volume of the prism?

5. A cube has an edge length of 2.5 ft. What is its volume?

6. A box measures 18 cm by 25 cm by 6 cm. What is the volume of the box?

7. The area of the base of a rectangular prism is 40 m^2. Its volume is 120 m^3. What is the height of the prism?

8. A cube has a volume of 1728 ft^3. Find its length, width, and height.

Now try these:

Lesson 7.4.2 additional questions — p467

Round Up

The formula $V = Bh$ is one you should learn, because you don't just use it with rectangular prisms. As you'll see in the next couple of Lessons, it's handy for finding the volumes of other 3D figures too.

Lesson 7.4.3

Volumes of Triangular Prisms and Cylinders

The formula you used last Lesson, $V = Bh$, doesn't just allow you to find the volume of a rectangular prism. In this Lesson, you'll be using it to find the volumes of triangular prisms and cylinders.

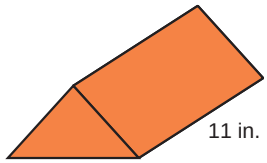
$V = Bh$ Can Be Used with Triangular Prisms

The formula you used to find the **volume** of a rectangular prism is $V = Bh$.

You can use **the same formula** to find the volume of a **triangular prism**.

Example 1

Find the volume of this triangular prism.
The area of its base is 45 in^2.



Solution

You can find the volume using the formula $V = Bh$

$V = Bh$

$V = 45 \times h$ Replace B with area of the base

$V = 45 \times 11$ Height of prism is distance between bases

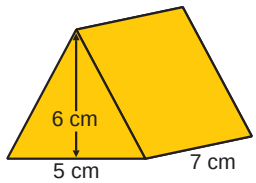
$V = 495$

So the volume of the prism is 495 in^3.

Sometimes you'll need to work out B first.

Example 2

Find the volume of this triangular prism.



Solution

You need to find the area of the triangular base first.

$A = \frac{1}{2}bh$

$A = \frac{1}{2} \times 5 \times 6 = 15$ cm^2

You can now find the volume using the formula $V = Bh$

$V = Bh$

$V = 15 \times 7$

$V = 105$

So the volume of the prism is 105 cm^3.

California Standard:

Measurement and Geometry 1.3
Know and use the formulas for the volume of triangular prisms and cylinders (area of base × height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.

What it means for you:

You'll use the formula $V = Bh$ to work out the volumes of triangular prisms and cylinders.

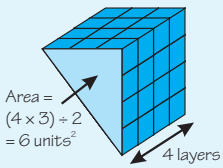
Key words:

- volume
- triangular prism
- cylinder
- base

Don't forget:

In the formula $V = Bh$:
- B is the area of one of the bases of the prism
- h is the height of the prism, or the distance between the two bases.

Check it out:



The formula is the same as the one from the previous Lesson, because, again, the volume is the number of unit cubes in one layer multiplied by the number of layers.

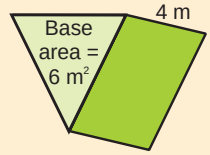
Check it out:

The formulas for area of a triangle and volume of a prism both have a lowercase h in them. One is the height of the triangle and one is the height of the prism.
Don't get them mixed up.

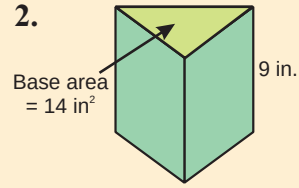
Guided Practice

Find the volume of the triangular prisms shown in Exercises 1–6.

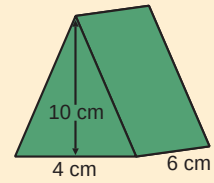
1.



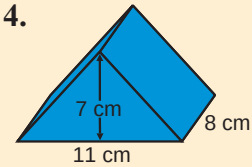
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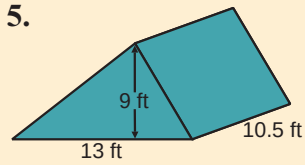
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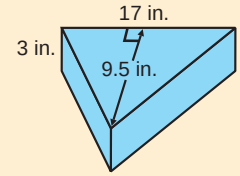
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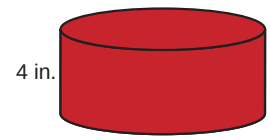


You Can Find the Volume of a Cylinder Using $V = Bh$

The formula $V = Bh$ even works for finding the volume of a **cylinder**.

Example 3

Find the volume of this cylinder.
The area of the base is 50.24 in^2.



Solution

You can find the volume using the formula $V = Bh$

$V = Bh$

$V = 50.24 \times h$ Replace B with area of the base

$V = 50.24 \times 4$ Height of prism is distance between bases

$V = 200.96$

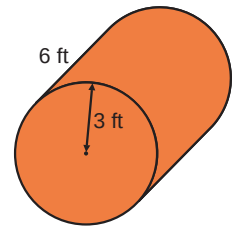
So the volume of the prism is **200.96 in^3**.

Don't forget:
Evaluate exponents before doing any multiplication.

Again, you might need to work out B first.

Example 4

Find the volume of this cylinder. Use 3.14 for π.



Solution

You need to find the area of the circular base first.

$B = \pi r^2$

$B = 3.14 \times 3^2$ Put in values for π and r

$B = 28.26$ ft^2

You can now find the volume using $V = Bh$

$V = Bh$

$V = 28.26 \times 6 = 169.56$

So the volume of the cylinder is **169.56 ft^3**.

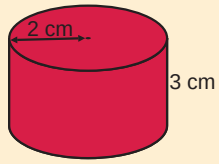
Don't forget:
See Lesson 7.1.3 for how to find the area of a circle.

Check it out:
The formula for the volume of a cylinder is often written
$V = \pi r^2 h$

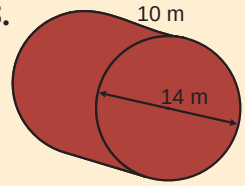
Guided Practice

In Exercises 7–12, find the volume of each cylinder. Use 3.14 for π.

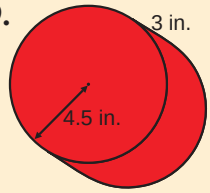
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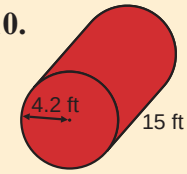
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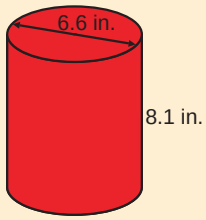
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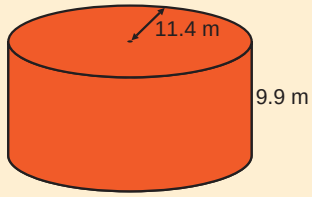
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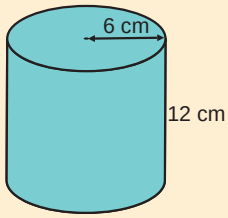


Don't forget:
You need the radius of a circle to find its area. If the question gives the diameter instead, then you need to divide that by 2 to get the radius.

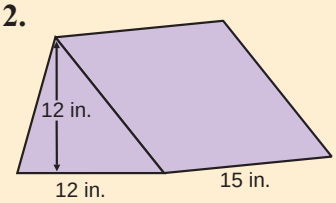
Independent Practice

In Exercises 1–6, find the volumes of the triangular prisms and cylinders shown. Use 3.14 for π.

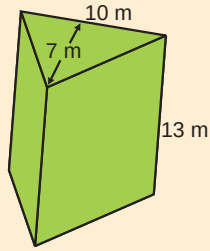
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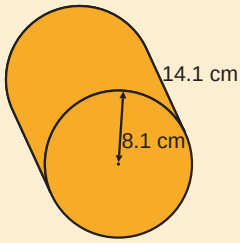
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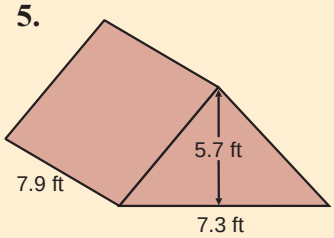
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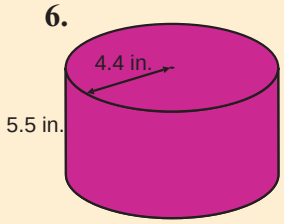
4.



5.



6.



7. The volume of a triangular prism is 248 in^3. The height of the prism is 8 in. What is the area of the base?

8. A swimming pool company makes a 15 ft diameter circular pool that is available in 3 ft, 3.5 ft, and 4 ft depths. Find the approximate number of cubic feet of water needed to fill each pool.

9. A candy company has decided to change the shape of its box. The old box was a rectangular prism with $l = 12$ cm, $w = 6$ cm, and $h = 2$ cm. Instead they will make a triangular prism in which the base of the triangle is 6 cm, and the height of the triangle is 12 cm. The height of the prism is 2 cm. How does this new design change the volume of the box?

Now try these:
Lesson 7.4.3 additional questions — p467

Round Up

This Lesson shows how useful the formula $V = Bh$ can be. If you learn the formula, and how to use it, you can work out the volume of any prism so long as you can find the area of the base.

Lesson 7.4.4

Volumes of Compound Solids

California Standard:

Measurement and Geometry 1.3
Know and use the formulas for the volume of triangular prisms and cylinders (area of base × height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.

What it means for you:
You'll use the formula $V = Bh$ to figure out the volumes of more complicated 3D figures.

Key words:
- volume
- rectangular prism
- triangular prism
- cylinder
- base
- area
- height

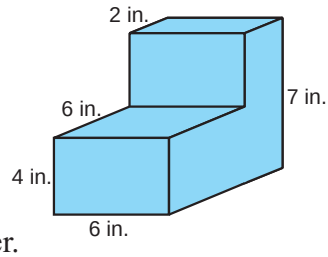
Check it out:
This is the 3D equivalent of the material in Lesson 2.3.3.

Now you know how to find the volume of rectangular prisms, triangular prisms, and cylinders. Sometimes you'll want to find the volume of solid shapes that look complicated, but can be broken down into smaller, simpler shapes.

Sometimes 3D Figures Need to Be Broken Down

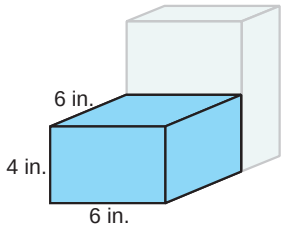
Example 1

Find the volume of this figure.



Solution

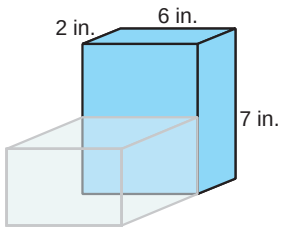
The figure is made up of two **rectangular prisms**. All you need to do is figure out the volume of each prism, then **add** the two together.



The volume of the first prism is

$$V = Bh$$
$$V = (l \times w) \times h$$
$$V = 6 \times 4 \times 6$$
$$V = \mathbf{144\ in^3}$$

The volume of the second prism is



$$V = Bh$$
$$V = (l \times w) \times h$$
$$V = 6 \times 7 \times 2$$
$$V = \mathbf{84\ in^3}$$

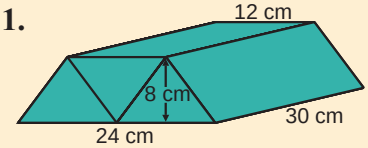
So the total volume of the figure is

$$144\ \text{in}^3 + 84\ \text{in}^3 = \mathbf{228\ in^3}$$

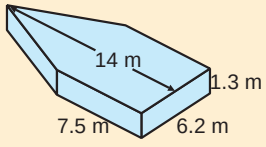
Guided Practice

In Exercises 1–5, find the volume of each figure. Use 3.14 for π.

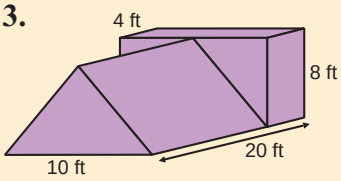
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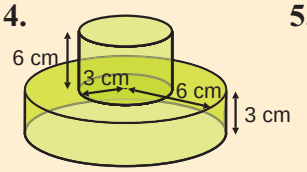
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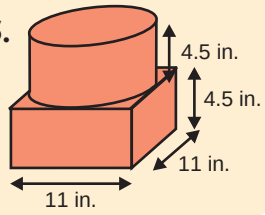
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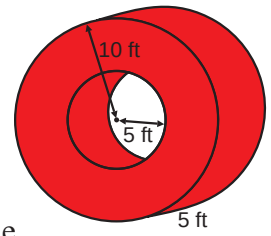
5.



Some 3D Figures Have Holes in Them

Example 2

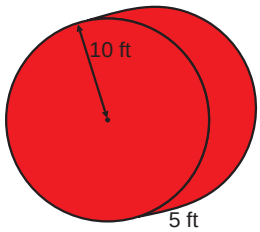
Find the volume of this figure. Use 3.14 for π.



Solution

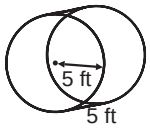
The figure is a **cylinder** with a **cylinder-shaped hole** through the middle. You first need to find the volume that the cylinder would have without the hole. Then you need to find the volume of the hole and take that away.

The volume of the full cylinder is



$$V = Bh$$
$$V = \pi r^2 \times h$$
$$V = 3.14 \times 10^2 \times 5$$
$$V = \mathbf{1570\ ft^3}$$

The volume of the hole is



$$V = Bh$$
$$V = \pi r^2 \times h$$
$$V = 3.14 \times 5^2 \times 5$$
$$V = \mathbf{392.5\ ft^3}$$

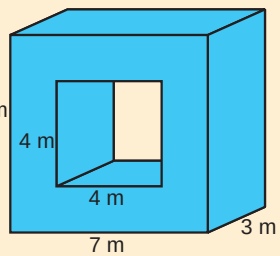
So the total volume of the figure is

$$1570\ \text{ft}^3 - 392.5\ \text{ft}^3 = \mathbf{1177.5\ ft^3}$$

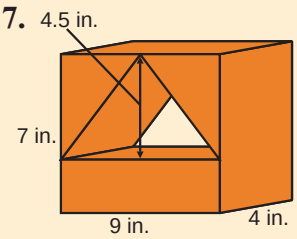
Guided Practice

In Exercises 6–9, find the volume of each figure. Use 3.14 for π.

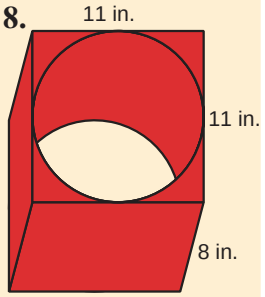
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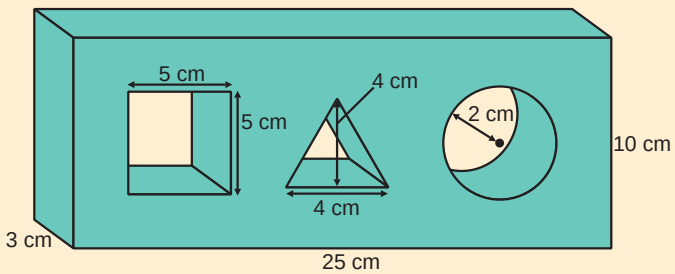
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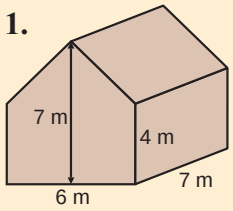
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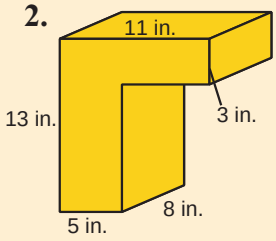
Independent Practice

In Exercises 1–6, find the volume of each figure. Use 3.14 for π.

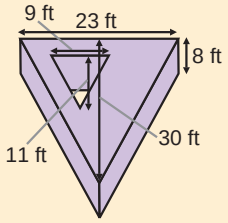
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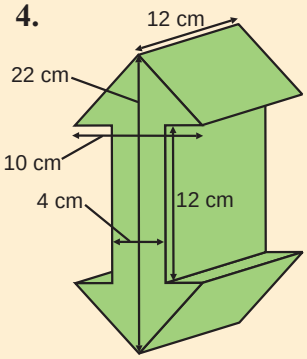
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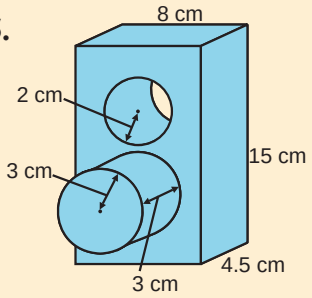
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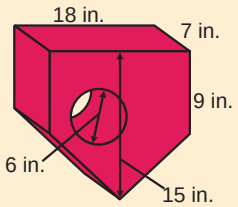
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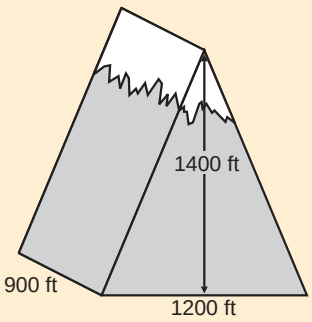


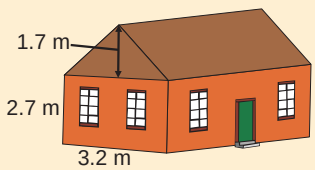
6.



7. A cylinder-shaped bottle of aspirin is packaged inside a box that is a square-based prism. The bottle has radius 2 cm and height 6.6 cm. The box has length 4.5 cm, width 4.5 cm, and height 7 cm. How much empty space is left inside the box once the bottle has been put in it?

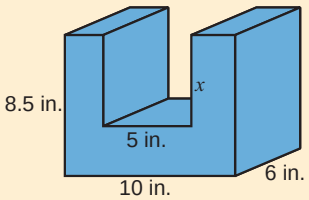
8. Mount Prism is a mountain that is shaped like a triangular prism. The size of the mountain is shown in the diagram. A tunnel is built straight through the mountain from one triangular side to the other. The tunnel is circular, with a diameter of 100 ft. What is the approximate remaining volume of the mountain once the tunnel has been dug out?





9. This building has a volume of 51.12 m^3. What is the length of the longer side?

10. This figure has a volume of 360 in^3. How long is the distance marked x?



Don't forget:
If there isn't a picture in the question, it usually helps to draw one for yourself.

Don't forget:
For questions involving real-life situations, you can take the value of π to be 3.14.

Now try these:
Lesson 7.4.4 additional questions — p468

Round Up

If you need to find the volume of a tricky-looking 3D figure, you should always try to split it up into simpler shapes. Remember to look out for any cases where it's easier to take a shape away rather than add them together.

Lesson 7.5.1

Section 7.5 Generalizing Results

There are lots of rules and patterns in math. In this Lesson, you'll learn to think about patterns of numbers in a way that will help you to figure out rules of your own.

To Find Patterns You Need to Collect Information

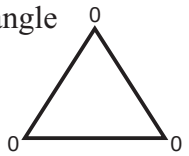
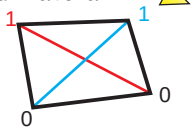
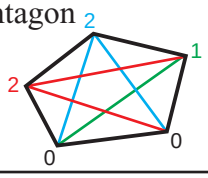
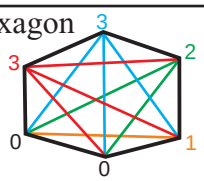
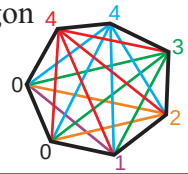
Often in math, you can **collect data** and look for **patterns**. This will help you make **predictions** and develop **rules**.

To look for a pattern, it helps to **record** your data in an **organized** way.

Example 1

Make a table of the number of diagonals in polygons of 3 to 7 sides.

Solution

Polygon	Number of sides	Number of diagonals
Triangle  The number at each vertex shows how many new diagonals start there.	3	$0 + 0 + 0 = 0$
Quadrilateral 	4	$1 + 1 + 0 + 0 = 2$
Pentagon 	5	$2 + 2 + 1 + 0 + 0 = 5$
Hexagon 	6	$3 + 3 + 2 + 1 + 0 + 0 = 9$
Heptagon 	7	$4 + 4 + 3 + 2 + 1 + 0 + 0 = 14$

California Standards:

Mathematical Reasoning 1.2 Formulate and justify mathematical conjectures based on a general description of the mathematical question or problem posed.

Mathematical Reasoning 3.1 Evaluate the reasonableness of the solution in the context of the original situation.

Mathematical Reasoning 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.

Mathematical Reasoning 3.3 Develop generalizations of the results obtained and the strategies used and apply them in new problem situations.

What it means for you:

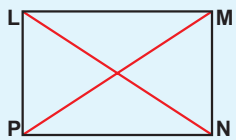
You'll see how to use information you have collected to find a rule and make predictions.

Key words:

- rule
- pattern
- prediction
- generalization
- formula

Check it out:

The examples in this Lesson are all about diagonals of polygons.



A diagonal of a polygon is a line joining two vertices that aren't already joined by the sides of the polygon. A rectangle has two diagonals.

Guided Practice

The numbers represented by this pattern of dots are called square numbers. Exercises 1–2 are about the pattern.



1. Copy and complete the table below.

Place in the pattern	Number of dots
1st	1
2nd	4
3rd	
4th	

2. Draw the 5th and 6th figures in the pattern. Add the number of dots in the 5th and 6th figures to the table.

You Can Make Rules and Predictions

When you've **collected** your information, you can look for **patterns** in the numbers. If you can figure out how the pattern will **continue**, then you can make a **rule**.

Example 2

Find the number of diagonals in an octagon (8-sided polygon).

Solution

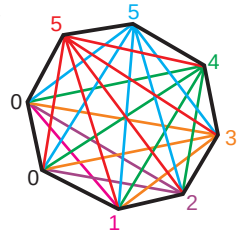
You need to look for a **pattern** in the **sums** used to find the number of diagonals in Example 1.

For each shape, the number of diagonals that can be drawn from the **first vertex** is 3 less than the number of sides. This number is **repeated once**, then **decreased by one** until 0 is reached on the last two vertices.

So an 8-sided polygon will have

$$5 + 5 + 4 + 3 + 2 + 1 + 0 + 0 = \textbf{20 diagonals}.$$

You can **check** this with a sketch.



Check it out:

The sketch for an 8-sided polygon is pretty confusing to look at and complicated to draw. For polygons with large numbers of sides, it would be almost impossible to draw a sketch. That's why it would be useful to find a rule, so you could work out the number of diagonals without having to draw a sketch.

Example 3

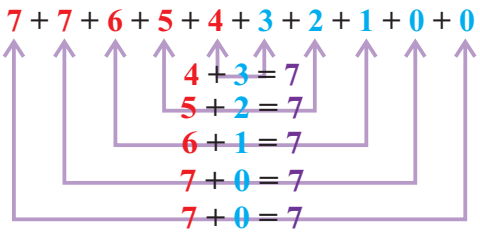
Find the number of diagonals in a 10-sided polygon.

Solution

You can find the answer using the same pattern as in Example 2:

$$7 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 + 0$$

There is a way to add the numbers up quickly — you can match the numbers into **pairs** that add up to **7**, the first number in the sum.



Each number in the sum represents **one vertex**. There are **10 vertices**, so there are $10 \div 2 = $ **5 pairs** that each add up to 7.

So there are $5 \times 7 = $ **35 diagonals** in a 10-sided polygon.

Check it out:

With an odd number of vertices, you can still pair up the numbers in the sum. There will be one number left unpaired. The unpaired number is always half the total of each pair of numbers. When you add this half-pair to the others, the number of pairs is still half the number of vertices.
Try this for yourself, using the heptagon from Example 1.

Example 4

Find a rule for working out the number of diagonals in a polygon with n sides.

Solution

You can say the same things for a polygon with ***n* sides** as you said for a polygon with 10 sides in Example 3.

Don't forget:

The letter n is a variable — the letter is just standing in for a number you don't know yet.

Number of vertices	Number of diagonals from first vertex	Number of pairs of vertices
10	$10 - 3$	$10 \div 2$
n	$n - 3$	$n \div 2$

For any ***n*-sided polygon**, we can **generalize** that the number of diagonals is always

$$(n - 3) \times (n \div 2)$$

Guided Practice

Exercises 3–5 use the table you made in Exercises 1–2.

3. Study the pattern of the numbers of dots. Predict how many dots would be in the 10th figure.

4. How could you predict the number of dots in the 100th figure?

5. Which of the following could be used as a general rule for finding the number of dots in the nth figure of the pattern?

$2n$ $\quad$ $n + 3$ $\quad$ n^2 $\quad$ $(n - 1)(n - 1)$

You Need to Test Rules to Make Sure They Work

When you figure out a new rule, you need to **test it** to see if it is true.

Example 5

Test the rule for working out the number of diagonals in any polygon.

Solution

The rule from Example 4 is that for any n-sided polygon, you can generalize that the number of diagonals is always $\mathbf{(n - 3) \times (n \div 2)}$.

You can test the generalization on polygons for which you **already know the results**.

The table from Example 1 says a hexagon has **9 diagonals**. If the rule works, it will give the answer 9 when $\mathbf{n = 6}$.

$$(n - 3) \times (n \div 2) \qquad \textbf{Write out the expression}$$
$$= (6 - 3) \times (6 \div 2) \qquad \textbf{Replace } n \textbf{ with 6}$$
$$= 3 \times 3 \qquad \textbf{Simplify}$$
$$= \mathbf{9}$$

The rule gives the **right answer** of **9 diagonals**.

Don't forget:
You could check that the rule works for all the polygons you've looked at so far — just to make sure.

Guided Practice

Exercises 6–7 use the rule you found in Exercises 3–5.

6. Test the rule you chose in Exercise 5 by checking that it gives the same answers as in the table from Exercises 1–2.

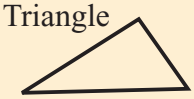
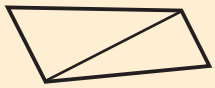
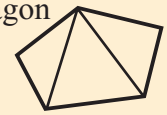
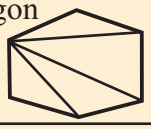
7. Use the rule to predict how many dots would be in the 7th figure in the pattern.

8. Draw the 7th figure in the pattern and count the dots to see if your answer to Exercise 7 was right.

Independent Practice

The measures of the angles in a triangle always add up to 180°. You can use this fact to make a generalization about the sum of the measures of the angles in any polygon.

1. In the chart below, diagonals are drawn from one vertex of each polygon to divide it into triangles. Copy and complete the chart.

Polygon	Number of sides	Number of triangles	Sum of measures of the angles
Triangle 	3	1	$1 \times 180° = 180°$
Quadrilateral 	4	2	$2 \times 180° = 360°$
Pentagon 	5	?	$3 \times 180° = ?$
Hexagon 	?	?	$? \times 180° = ?$

2. Continue your chart for a heptagon (7-sided polygon) and an octagon (8-sided polygon). Remember to draw diagonals from one vertex only to break the polygon into nonoverlapping triangles.

3. Without drawing it, predict the number of triangles and the sum of the measures of the angles in a 10-sided polygon.

4. Study your chart. What is the relationship between the number of sides and number of triangles in a polygon?

5. For an 85-sided polygon, which of the following could be used to find the sum of the measures of its angles?

$83 \times 180°$ $84 \times 180°$ $85 \times 180°$ $86 \times 180°$ $87 \times 180°$

6. Describe in words a rule that could be used for finding the sum of the measures of the angles in any polygon.

7. Write a formula for finding the sum of the angle measures in an n-sided polygon. Use S to represent the sum of the angle measures. It may help to first complete each column of your table for an n-sided polygon.

Now try these:
Lesson 7.5.1 additional questions — p468

Round Up

That might seem like a lot to take in, but the Lesson boils down to three basic steps for finding math rules. First, collect and organize your data. Next, look for a pattern and use it to find a rule. Finally, test the rule to make sure it works.

Lesson 7.5.2

Proving Generalizations

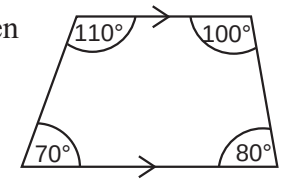
When you make a generalization, you might not be able to tell if it's always true. It isn't true if you can find even one example where the rule doesn't work. If it is true, there's always a mathematical reason why.

Generalizations Can Sometimes Be Proved Wrong

You might find some information that leads you to make a **generalization**. But sometimes you'll then be able to find an example that **proves** the generalization **isn't true**.

Example 1

Tara measured the angles of this trapezoid. She then made a generalization that no two angles in a trapezoid ever have the same measure.

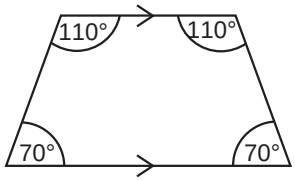


Give an example of a trapezoid that shows that this generalization is incorrect.

What was wrong with Tara's method of making a generalization?

Solution

The trapezoid shown here is **one example** that proves that Tara's generalization is wrong. You could show any **isosceles** or **right** trapezoid.



Tara based her generalization on **only one example**. If she'd looked at **more** trapezoids, she might have found one with two angles of the same measure. She didn't **test** if she could draw a trapezoid with more than one angle of equal measure.

Guided Practice

In Exercises 1–5, use an example to prove that each generalization is not true.

1. In a right triangle, the two angles that are not right angles always have the same measure.

2. A trapezoid can never contain a right angle.

3. If two sides of a triangle are the same length, the third side is always a different length.

4. Rectangles with different side lengths always have different areas.

5. The two sides of equal length in an isosceles triangle are each always longer than the other side.

California Standards:

Mathematical Reasoning 1.2 Formulate and justify mathematical conjectures based on a general description of the mathematical question or problem posed.

Mathematical Reasoning 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.

Mathematical Reasoning 3.3 Develop generalizations of the results obtained and the strategies used and apply them in new problem situations.

What it means for you:

You'll find out about using examples to prove when generalizations are not true, and about using math to prove when they are.

Key words:

- generalization
- prove
- example
- test

Don't forget:

A trapezoid is a quadrilateral with only one pair of parallel sides.

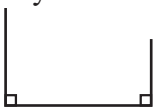
You Can Prove a Generalization Using Math

Example 2

Is it possible to draw a triangle with two right angles?

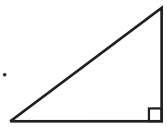
Solution

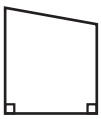
If you can draw a triangle with two right angles, then the answer is yes.



This attempt has **failed** because the shape **doesn't close**.

This attempt has **failed** because there's only **one right angle**.





This attempt has **failed** because there are **four sides**.

It looks like you can make the following **generalization**:
"it's **impossible** for a triangle to contain **two right angles**."
But there's no way to tell for sure if that's true by drawing examples — there might be a way to do it that you just haven't thought of.

Check it out:
If you're using examples, you only need one to prove that something isn't true.
But proving that something is true using examples can be tricky.

The way to show a generalization is true is to start with some **known facts**.

Example 3

Use the triangle sum to show that a triangle can't have two right angles.

Solution

A right angle measures **90°**.

If a triangle has **two** right angles, the sum of their measures is

$$90° + 90° = \mathbf{180°}.$$

For the sum of **all three angles** to be 180°, the third angle would have to measure **0°**. But a triangle **can't** contain an angle of 0°, so a triangle **can't** have two right angles.

Don't forget:
The triangle sum tells us that the measures of the angles in a triangle add up to 180°.

Guided Practice

In Exercises 6–9, try to draw a quadrilateral with the following characteristics.

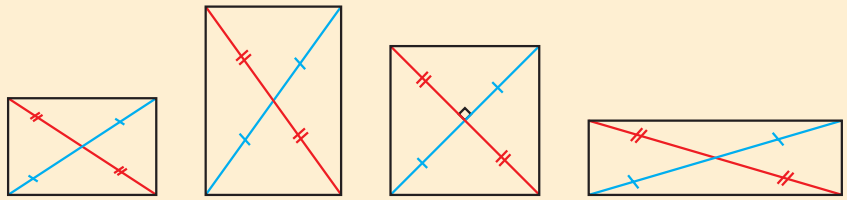
6. Exactly 1 right angle
7. Exactly 2 right angles
8. Exactly 3 right angles
9. Exactly 4 right angles

10. Use math to prove the generalization: "It is not possible to draw a four-sided shape that contains exactly three right angles."

Independent Practice

Don't forget:
Matching tick marks show line segments of equal length.

There are two diagonals that can be drawn in every rectangle. A few examples are shown below.



From looking at the examples above, say whether the following generalizations about the diagonals of all rectangles are true.

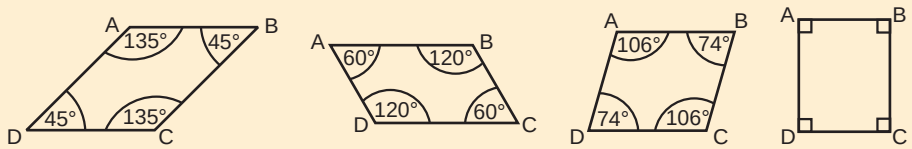
1. The two diagonals are always the same length.
2. The two diagonals cut each other into equal-length segments.
3. The diagonals intersect at a 90° angle.

Don't forget:
"Rhombi" is the plural of "rhombus." See Lesson 7.3.3 for more information.

In Exercises 4–7, draw several examples of the given type of quadrilateral. Determine which of the generalizations from Exercise 1–3 are true for all quadrilaterals of that type.

4. Squares
5. Rhombi
6. Parallelograms
7. Trapezoids

Use the parallelograms shown below to answer Exercises 8–13.



8. Compare the measures of angles A and C in each parallelogram. What generalization can you make about the opposite angles A and C in the parallelograms above?

9. Is this generalization true for opposite angles B and D in each parallelogram above?

Now try these:
Lesson 7.5.2 additional questions — p469

10. Is it possible to draw a parallelogram where these generalizations are not true? If so, draw it. If not, explain why not.

11. For each parallelogram, find the sum of angles A and B.

12. What generalization can you make about the neighboring angles A and B in the parallelograms above?

13. There are other pairs of neighboring angles in each of the parallelograms. Is your generalization in Exercise 12 true for the pairs B and C, C and D, and D and A as well?

Round Up

Sometimes you can prove a generalization using examples, but usually you'll need to use the math behind the generalization to show that it is always true. That doesn't mean that examples aren't important — they're often the best way to prove when a generalization isn't true.

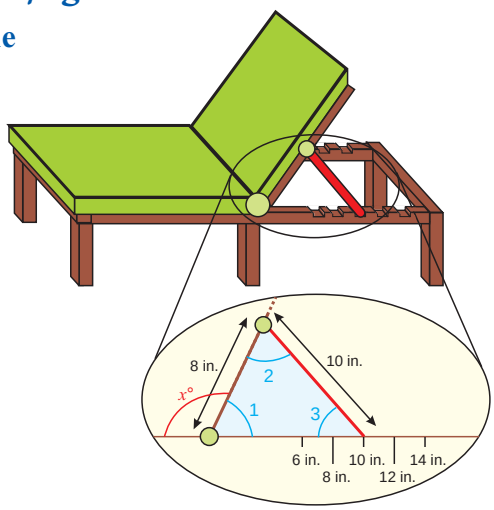
Chapter 7 Investigation

Reclining Chairs

Reclining lounge chairs at hotel pools can usually be set to recline at various angles. The support bar at the back of the seat is moved into designated slots — like in the picture below. The nearer the slot to the base of the seat, the more upright the back will be.

When viewed from the side, the **support bar**, the **back of the seat**, and the **slotted bar** form a **triangle**, as shown.

- The support bar is **10 inches** long.
- It is permanently attached to the back of the chair at a length of **8 inches** from the hinge of the seat.
- The support bar can be moved into slots at **6, 8, 10, 12, or 14 inches** from the hinge of the seat to create different reclining positions. It can also be laid flat.



Part 1:

How is the **reclining angle $x°$** related to **angle 1**?

How is angle 1 affected when the support brace is moved into **different slots**?

Part 2:

Investigate the **different types of triangles** that can be created by the support bar, the back of the seat, and the slotted bar. Use a compass and ruler to draw the triangles.

Extension

The reclining angle ($x°$) has a special relationship to the other angles in the "support triangle."

- Describe any patterns you notice between x and the measures of angles 1, 2, and 3. Try to make a generalization about any relationships you see.
- Test your generalization for other triangles. Is this relationship true of all triangles? If not, explain why not. If so, generalize your findings.

Open-ended Extension

Create a design for a lounge chair different than the model given.

- What reclining angles would your chair have?
- Could you use the same basic design?

 If so, what length would the chair back and support bar be? Why?
 If not, explain why not, and design an alternative reclining mechanism.

Round Up

This is just the kind of thing some engineers have to do. They're given a specification — a description of something that someone needs, giving information about what it must do. The engineer must then come up with a design that will do all of the things necessary.

Additional Questions

Lesson 1.1.1 — Comparing Integers

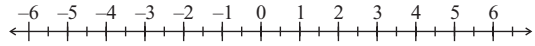
In Exercises 1–3, say which of the integers is greater.

1. –4, 4 **2.** 5, –6 **3.** –2, –4

In Exercises 4–6, list all the numbers from –3 through 4 that are:

4. whole numbers **5.** integers **6.** natural numbers

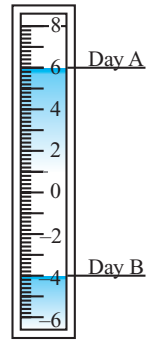
The change in the position of a football after Timmy ran it was +2 yards. The last time Jamal ran the football the change of position was –4 yards.



7. Circle these two integers on a number line

8. Use the correct symbol, "<", ">", or "=", to compare these two integers.

When the water level in a hotel pool is below the 0 marker, Andrew needs to add more water. When it is at or above 0 he does not. The water level during two different days is shown on the water meter to the right.



9. Write all the integers between the two levels of water shown.

10. Describe the range of numbers shown on the water meter at which Andrew does not need to add water as natural numbers, whole numbers, or integers.

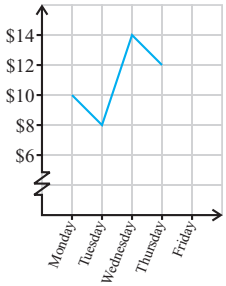
Lesson 1.1.2 — Adding and Subtracting Integers

In Exercises 1–6, work out the result of each calculation.

1. 8 + 4 **2.** 6 + (–5) **3.** –12 + (–3)

4. –3 – 2 **5.** –14 – (–8) **6.** –9 – (–6)

7. How is subtracting a negative number using the number line different from adding a negative?



The graph shows the price of a stock over four days. Use this for Exercises 8–10.

8. Write the integer that represents the change in the price of the stock between Monday and Tuesday.

9. Write the integer that represents the change in the price of the stock between Tuesday and Wednesday.

10. The stock price changed by –$3 overall from Monday to Friday that week. Write the price of the stock on Friday.

Janet played four rounds of golf in a tournament. Her scores for the first three rounds are shown in the table. Use this for Exercises 11–13.

11. What was Janet's overall score after round 2?

12. Janet's overall score for all four rounds was –7. What was Janet's score in round 4?

13. Janet's overall score after four rounds was 2 lower than Melissa's. Write Melissa's score for round 4.

Golf Scores

Player	Rd 1	Rd 2	Rd 3	Rd 4
Janet	–3	–1	2	
Melissa	–4	1	–1	

Lesson 1.2.1 — Multiplying with Integers

In Exercises 1–3, work out the result of each calculation.

1. –6 × 8 **2.** –3 × (–5) **3.** –12 × (–3) × (–2)

4. Explain why these two expressions have the same value: 6 × 3 and 3 × 6

A cafeteria used five cans of soup every day, for eight days. To work out the change in the number of soup cans in stock, a worker wrote this expression: –5 × 8.

5. Write the value of this expression.

6. The number of cans of soup left at the end of the eight days was 164. How many cans of soup did the cafeteria have at the beginning of the eight days?

Lesson 1.2.2 — Dividing with Integers

In Exercises 1–3, work out the result of each calculation.

1. $72 \div (-9)$ **2.** $-24 \div 3$ **3.** $-90 \div (-5)$

Mary did the division, $54 \div (-2)$ and got the answer -26.

4. Describe how Mary could check her result without repeating the division.

5. Say whether -26 is the correct answer to the division. If it isn't, find the correct answer.

6. Write an expression with a quotient of -12.

A worker at a laundromat needed to do 15 basketfuls of laundry for a hotel client. Each basket had 10 pounds of laundry in it. The washers could hold 25 pounds of laundry each.

7. Write the total number of pounds of laundry the worker needed to do.

8. Write the least number of washes the worker could do to complete the order.

Malcolm is building a fence. He plans to use 18 fence posts, and to put a topper on every third post.

9. How many toppers will Malcolm need?

10. The fence posts are 3 feet high without the toppers. The toppers are $\frac{1}{3}$ the height of the original fence posts. What will be the height of a fence post with a topper added?

Lesson 1.2.3 — Integers in Real Life

In Exercises 1–4, work out the result of each calculation.

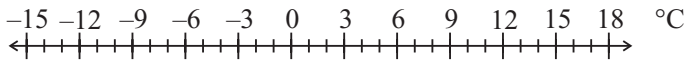
1. 1.5 miles multiplied by four.

2. 100 people divided into four equal groups

3. 150 feet, dropping by 40 feet

4. \$20 plus \$134

In Exercises 5–6, use the number line below.



5. Write an integer to show the temperature change if a thermometer reading goes from –1 °C to –8 °C.

6. Write two integers from the number line with a difference of 21 °C.

7. A deliveryman delivered 20 packages each hour for three hours. After this time he still had 70 packages left to deliver. If he carried on delivering packages at the same rate, how long would it take him in total to deliver all the packages?

Lesson 1.3.1 — Decimals

In Exercises 1–3, write the digit in the hundredths place.

1. 152.685 **2.** 565.642 **3.** 33.896

The changes in a gymnastic team's average score after each of four events are shown in the table to the right.

Event	Change in average score
Vault	+0.4
Uneven Bars	+0.1
Balance Beam	
Floor	

4. The change in the average score after the balance beam was an increase of 3 tenths. Represent this change using a decimal.

5. The change in the average score after the floor exercise was a drop of 5 tenths. Represent this change using a decimal.

Package weight	Mailing cost
< 0.5 lb	\$2.00
0.5 to 0.999 lb	\$4.00
1.0 to 1.499 lb	\$6.00
1.5 to 1.999 lb	\$8.00

Suzanne worked in a mail room. She found the weight of several packages and charged the customers based on that weight as shown in the table to the left.

6. Suzanne weighed a package that was between 0.6 and 0.7 pounds. Find the mailing cost.

7. Suzanne then weighed another package. The cost of mailing the first package (in Exercise 6) and the second package together was \$12.00. Write a possible weight for the second package.

Lesson 1.3.2 — Ordering Decimals

In Exercises 1–2, place the numbers in order of increasing value.

1. –2.67, –0.67, –1.275, 0, –5

2. 0.1, –0.03, –3.2, –3.26, 0.15

3. Explain how to add zeros to compare the decimal numbers 4.52 and 4.5.

After a rainstorm a class measured three puddles on the school playground.
The width of the first puddle was 21.4 cm.

4. The width of the second puddle was less than the width of the first, but greater than 20 cm. Write a decimal number that could represent the width of the second puddle in centimeters.

5. The third puddle was more than two times wider than the second puddle. Write a decimal number that could represent the width of the third puddle in centimeters.

Lesson 1.4.1 — Rounding Numbers

In Exercises 1–3, write the numbers –1001.55, –479.133, and 49.806, to the nearest:

1. one

2. hundred

3. hundredth

4. A scientist showed that the average raindrop measures 0.001655 meters across. What is this to the nearest thousandth of a meter?

5. When Marsha rounded three numbers to the nearest hundredth, she got a result of 0.53 all three times. Two of the numbers were greater than 0.53 and one was less. Write three numbers that could have been the numbers that Marsha rounded.

Country	Area (km^2)
Russia	8,502,290
Brazil	5,326,884
Canada	2,456,996
United States	2,262,039
China	1,629,600

The table to the left shows the forested area of various countries in the world. Use the table to answer Exercises 6–8.

6. Write the forested area in Russia to the nearest hundred thousand square kilometers.

7. Write the forested area in Canada to the nearest ten thousand square kilometers.

8. When the forested area of the United States was rounded, it came to 2,000,000 km^2. What place value was the forested area rounded to?

Lesson 1.4.2 — Using Rounded Numbers

In Exercises 1–3, use estimation to find approximate answers

1. 518 + 287

2. 16,994 – 5527

3. 3889 × 689

4. Use the sum 847 + 145 to explain how the place value to which a number is rounded affects the accuracy of the estimation.

The table to the right lists the average snowfall in five places in the US. Use this information in Exercises 5–7.

5. List the average snowfalls from the chart, when rounded to the nearest 10.

6. Estimate the difference between the average snows in Valdez, Alaska and Marquette, Michigan.

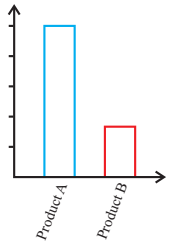
7. Estimate the total number of inches of snow the people in Yakutat, Alaska should expect during a 9 year period.

Place	Ave. inches snow/year
Valdez, Alaska	326
Mount Washington, NH	260
Blue Canyon, California	240
Yakutat, Alaska	195
Marquette, Michigan	141

8. The greatest TV audience for one program was 114,970,000 people. Write the greatest and least possible audience figures, given that this number is rounded to the nearest 10,000 people.

Lesson 1.4.3 — Estimation

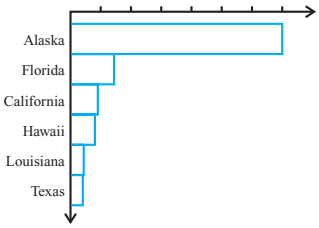
In Exercises 1–3 use the bar graph of sales during September to answer the questions.



1. Estimate the number of times greater the sales for Product A were, than for Product B.

2. Product B had sales of between 200,000 and 250,000. Estimate the sales for Product A

3. In the month of October, the sales for each product rose by $\frac{1}{4}$ of the amount they sold in September. Estimate the sales for each product in October.



The lengths of coastlines in several states are shown on the graph. Alaska has a coastline of 6640 miles. Use these facts to answer Exercises 4–6.

4. Write an estimate for the length of the Californian coastline. Explain your estimate.

5. Write an estimate of the coastline length in Texas. Explain your answer.

6. Write an estimate to describe the difference between the coastline lengths in Texas and Florida. Explain your estimate.

Lesson 1.4.4 — Using Estimation

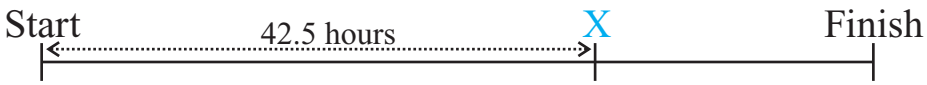
In Exercises 1–3, say whether a precise answer or an estimate is appropriate.

1. Coach Henderson is measuring the distance jumped by 10 students during a long-jump competition.

2. Pauline is deciding how much paint to buy to paint the outside of her house.

3. Mrs. Jones is making a cake and is measuring the amount of flour she needs.

A boat traveling down a river was located at the point marked X after 42.5 hours of travel.



4. Estimate the total amount of time it will take the boat to travel from start to finish. Explain your estimate.

5. The distance from start to finish was 360 miles. Estimate the distance the boat had left to travel.

6. Say whether or not the boat would be able to travel 500 miles in 100 hours. Explain your answer.

Lesson 2.1.1 — Variables

1. Gina has scored 47 more points this basketball season than Isabella. What letter would be a sensible variable to use to represent the number of points scored by Isabella.

In Exercises 2–7, show the expressions on a number line.

2. $a - 2$ **3.** $r - (-4)$ **4.** $y + (-6)$

5. $s \div 5$ **6.** $w \times 3$ **7.** $b \div (-2)$

In Exercises 8–11, determine whether or not it is necessary to use a variable to describe the problem.

8. Emilio has twice as many spelling awards as Jake. Determine the number of spelling awards that Jake has.

9. Mrs. Gooden planted 3 times the number of daisies that Mr. Kearns planted. Mr Kearns planted 12 daisies. Determine the number of daisies Mrs. Gooden planted.

10. A baker sold 36 more doughnuts than sweet rolls during one day. She wants to know how many doughnuts she sold.

11. Luke pitched 12 baseball games this season. Jose pitched one-third of the number of baseball games that Luke pitched. Their coach wants to know how many baseball games Jose pitched this season.

12. Write a problem in which the h in $h \div 4$ is an unknown quantity in a situation.

13. Write a situation in which the unknown is the number of movies Andre saw last summer.

Lesson 2.1.2 — Expressions

In Exercises 1–5, write an expression for each situation.

1. A number, p, is increased by 12.

2. A number, y, is multiplied by 7.

3. A number, h, is decreased by 36.

4. A number, w, is divided by 2.

5. A florist divided 64 roses equally between v vases. How many roses were in each vase?

Evaluate Exercises 6–8, if the value of a is 9.

6. $(a \times 2) + 13$

7. $55 - (a \div 3)$

8. $(9 \div 3) + a$

In Exercises 9–11, describe each expression in your own words.

9. $(j \times 9) + 5$

10. $23 + (d \div 2)$

11. $15 - (3 \times e) + 1$

12. The cost of bananas was $\$0.32p$, where p is the number of pounds of bananas sold. How much would 4 pounds of bananas cost?

13. Write an expression in which d is multiplied by 5, and then the product is decreased by 12.

14. Evaluate your expression from Exercise 13 for $d = 8$.

Lesson 2.1.3 — Multi-Variable Expressions

In Exercises 1–3, evaluate the expressions for $x = 1$ and $y = 2$.

1. $x + 2y$

2. $(6 - y) + (x \times 4)$

3. $5 \times (y \div x) + 3$

In Exercises 4–6, write an expression for each situation.

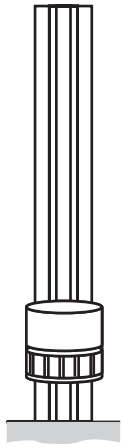
4. c is increased by the product of y and r

5. The ratio of j to k is decreased by the sum of w and z

6. $6m$ is decreased by the product of g, h, and i

7. There were red, green, and yellow footballs in a P.E. equipment box. There were three times as many red footballs as the difference between the number of green and yellow footballs. Write an expression for the number of red footballs.

Exercises 8–9 use the diagram to the right, which shows an amusement park ride that travels up and down.



8. The ride travels straight up to a maximum height of 150 feet off of the ground. From there it drops x feet, and goes up another y feet, then goes down the remaining z feet to the ground. Write an expression to show the total distance, in feet, a person on the ride would travel.

9. Find the total distance, in feet, a person on the ride would travel when $x = 70$, $y = 45$, and $z = 115$.

Lesson 2.1.4 — Order of Operations

1. Rewrite w^5 as a repeated multiplication of w.

2. Rewrite the expression $b \times b \times b \times b \times b \times b \times b \times b$ in base and exponent form.

3. What does PEMDAS stand for?

4. Explain the order of operations in the expression $15 + (2^3 \div 3 \times 4 - 1)$.

5. Ms. Vaughn is giving 15 questions to every student in each of her three math groups. There are 8 students in one math group, 7 in a second math group, and s students in the third math group. Write an expression that can be used to calculate the total number of questions Ms. Vaughn will have given out.

6. Given that there are 9 students in Ms. Vaughn's third math group, determine the value of your expression from Exercise 5.

Lesson 2.2.1 — Equations

1. What is the main difference between an equation and an expression?

2. Use the guess and check method to solve the equation $x - 12 = 10$.
Give 2 examples of a guess and check, then show the correct value of x.

In Exercises 3–5, write down the value of n that satisfies each equation.

3. $100 = 9n + 19$ **4.** $n \div 6 = 2$ **5.** $5n - 45 = 90$

In Exercises 6–9, write an equation that represents each given situation.

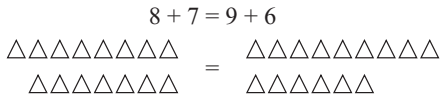
6. Aaron is 6 inches taller than Mario.
7. Hailey is 9 years younger than Cassandra.
8. Gavin ate half as many hot dogs as Lin.
9. Trina needs twice as much fabric as Bea.

10. A hairstylist makes \$25 per haircut and \$80 per color. Write an expression for the total amount of money the hairstylist makes for h haircuts and c colors.
11. On Saturday, the hairstylist gave 7 haircuts and 4 colors. Write an equation for the total amount of money, m, that the hairstylist made that day. Solve your equation for m.

Lesson 2.2.2 — Manipulating Equations

1. When you manipulate equations, what is the only way to keep both sides of the equation balanced?

2. The diagram to the right shows a balanced equation. If you add 5 triangles to the left side of the equation, what has to be done to the right side?



In Exercises 3–5, fill in the missing parts.

3. $d - c = 34$
$d - c + ? = 44$

4. $6y = 24$
$6y \div ? = 12$

5. $t - 1 = 64$
$? = 64 - 64$

In Exercises 6–8, explain what step was done incorrectly.

6. $a \times b = 130$
$a = 130 + b$

7. $q = 50$
$5q = 10$

8. $m - 9 = n$
$m + 2 = n - 11$

9. Ike saw that Jenna had 32 ¢ more than he did. Write an equation to represent this situation.

10. Using your answer from Exercise 23, if Jenna has 96 ¢, how many cents does Ike have?

Lesson 2.2.3 — Solving + and – Equations

Tomas has solved the equation $x + 12 = 20$ and found that $x = 8$.
1. Explain how he can check that this answer is correct.
2. Is Tomas's solution to the equation correct? Show your work.

Solve the equations in Exercises 3–11 using manipulation. Remember to check your answers.

3. $y + 5 = 13$
4. $2.2 + y = 5.3$
5. $-15 = y + 12$
6. $75 = y - 25$
7. $y + 20 = -20$
8. $y + -6 = -31$
9. $-17 = y - 9$
10. $-7 = -7 + y$
11. $y - 3.7 = 9.8$

In Exercises 12–15, write an equation for each situation, then solve it. Use n to represent the variable.

12. Three plus a number is 9.
13. A number decreased by 10 is equal to 91.
14. When five is added to a number the result is 52.
15. A number increased by 0.2 equals 1.8

16. Kelsey's plant has grown 3 inches since last Tuesday and is now 10 inches tall. Write an equation to represent this situation. Use h to represent the height of Kelsey's plant last Tuesday. Then solve your equation to find h.

Lesson 2.2.4 — Solving × and ÷ Equations

Write down the solution to the multiplication equations in Exercises 1–3.

1. $15a = 90$ **2.** $j \times 3 = 108$ **3.** $20 \cdot q = -480$

Write down the solution to the division equations in Exercises 4–6.

4. $12 \div h = 4$ **5.** $32 \div m = 2$ **6.** $g \div 5 = -9$

7. Ryan made a mistake in solving the equation to the right. What mistake did Ryan make in his work?

$3t = 27$
$t = 81$

8. Solve Ryan's equation correctly.

In Exercises 9–12, write an equation for each situation, then solve it. Use y to represent the variable.

9. A number multiplied by seven is equal to 21.

10. A number divided by –5 equals twenty.

11. The product of a number and –3 is eighteen.

12. Four divided by a number equals 16.

13. The Bobcats football team scored 18 points this year from 2-point safeties. Write and solve a multiplication equation that can be used to determine the number of safeties, s, that the Bobcats made this year.

Lesson 2.2.5 — Graphing Equations

1. What is a different way to write the equation $b \div 5 = 40$?

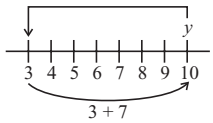
2. What are different ways to write the equation $m \times 6 = 22$?

3. How does the equation $x + 3 = 12$ explain the location of x on a number line?

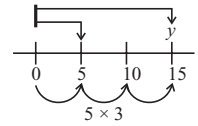
4. How does the equation $y - 2 = 7$ explain the location of y on a number line?

In Exercises 5–6, look at the diagram and write down the equation that goes with each.

5.



6.



In Exercises 7–11, identify the type of equation and solve it.

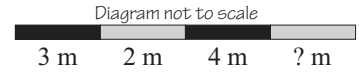
7. $k - 40 = 92$ **8.** $19j = -38$ **9.** $w + 14 = 65$

10. $a \div -7 = 11$ **11.** $n \times 25 = 200$

12. Coach Peters had 80 uniforms that are to be shared equally among x players. However, he lost 23 of the uniforms leaving each player with u uniforms. If there are 19 players, calculate how many uniforms each player receives by writing and solving a suitable equation.

Lesson 2.3.1 — Expressions About Length

1. Write down an expression that represents the length of this object.



2. The equation $(x + y) + z = x + (y + z)$ is an example of what property of addition?

3. Prove the equation using $x = 7$, $y = 4$ and $z = 9$.

In Exercises 4–6, write down a shape whose perimeter could be represented by the expression

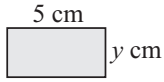
4. $5 + 6 + x$ **5.** $3 + 3 + 3 + 3 + x$ **6.** $9 + 12 + 4 + x$

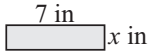
7. Malik is cutting balloon strings. The strings that came with the balloons are too long. Each string is i inches long and needs to be exactly 12 inches. Write an expression that could be used to represent the length of the string that has been cut off.

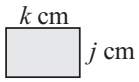
8. Miss Ortiz is arranging desks in her classroom. Each desk is rectangular in shape and has a length of 26 inches and a width of 16 inches. She arranged 6 desks so the widths of the desks were side-by-side. Find the perimeter of the row of 6 desks Miss Ortiz arranged.

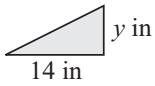
Lesson 2.3.2 — Expressions About Areas

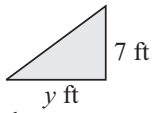
In Exercises 1–6, write an expression for the area of each shape.

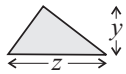
1. 

2. 

3. 

4. 

5. 

6. 

In Exercises 7–10, write an expression that represents the area of each rectangle described.

7. Base = 3 cm and Height = h cm

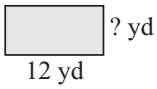
8. Base = d inches and Height = 12 inches

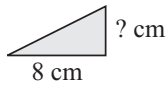
9. Base = m feet and Height = w feet

10. Base = 17 inches and Height = 11 inches

11. The equation $(xy)z = x(yz)$ is an example of what property of multiplication? Show an example using $x = 7$, $y = 4$, and $z = 9$.

In Exercises 12–13, find the length of the unknown side when the area of each figure is 48.

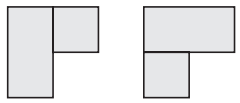
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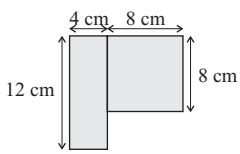
14. Aubrey wants to frame a picture that is 8 inches long by 10 inches wide. He wants to use a frame that is 16 inches long by 20 inches wide. Determine the area of the picture and the frame. Find the amount of space inside the frame that will not be occupied by the picture.

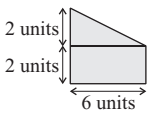
Lesson 2.3.3 — Finding Complex Areas

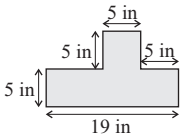
1. Dora and Ben divided the same complex shape differently to determine the area. Their drawings are shown to the right. Explain how Dora and Ben get the same answer for the area when they each divided the shape up differently.

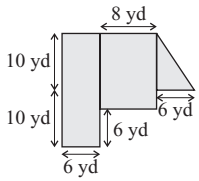


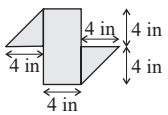
In Exercises 2–10, find the area, or an expression for the area, of each figure.

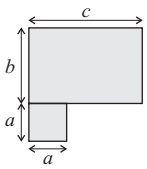
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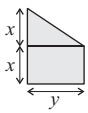
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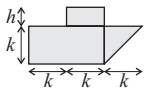
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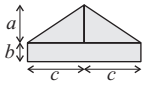
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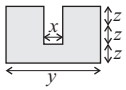
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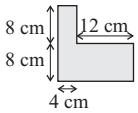
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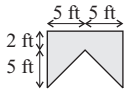
9. 

10. 

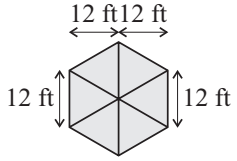
In Exercises 11–13, use subtraction to determine the area of each figure.

11. 

12. 

13. 

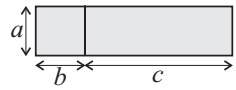
14. A gazebo is being built in a park. The floor of the gazebo is shaped like a regular hexagon, as pictured to the right. Determine the area of the floor of the gazebo. Explain, in words, how you determined your answer.



Lesson 2.3.4 — The Distributive Property

1. The equation $a(b + c) = ab + bc$ is an example of what mathematical property? Verify this property when $a = 4$, $b = 9$ and $c = 2$.

2. Verify the distributive property by finding the area of the figure to the right, in two different ways.



In Exercises 3–5, draw a picture to represent each expression.

3. $3 \times (4 + 5)$ **4.** $8 \times (6 + 7)$ **5.** $2 \times (9 + y)$

Use the distributive property in Exercises 6–8 to determine the value of each expression.

6. $3 \times (12 - 9)$ **7.** $-1 \times (13 + 27)$ **8.** $-5 \times (-6 - 9)$

Remove the parentheses from each expression in Exercises 9–11.

9. $k \times (g - d)$ **10.** $(5 - c) \times 2$ **11.** $(m - v) \times x$

Lesson 2.3.5 — Using the Distributive Property

1. Explain how the distributive property can help with mental math.

Show how you would use the distributive property to solve Exercises 2–7.

2. 7×15 **3.** 4×17 **4.** 18×6

5. 17×8 **6.** 14×9 **7.** 23×5

8. Deshaun had one dollar, made up of the following coins: 3 quarters, 2 dimes, and 1 nickel. The total number of coins is given by $3q + 2d + n$, where q is the number of quarters, d is the number of dimes, and n is the number of nickels. Write an expression for the number of coins Deshaun would have if he had 5 dollars made up of the same combination of coins given above.

Rewrite Exercises 9–11 using the distributive property.

9. $g \times (h + 19)$ **10.** $11 \times (4 - w)$ **11.** $a \times (b + d + f)$

12. A sheet of stickers contains dogs, cats, birds, and fish. The total number of stickers on each sheet is given by $3d + 5c + 8b + 4f$, where d is the number of dog stickers, c is the number of cat stickers, b is the number of bird stickers, and f is the number of fish stickers. Write two expressions for the total number of stickers on 4 of these sticker sheets. Evaluate your expression for $d = 2$, $c = 6$, $b = 3$, and $f = 5$.

Lesson 2.3.6 — Squares and Cubes

In Exercises 1–4, write an expression for the total area of each figure.

1. A square with sides of length m

2. Three identical squares with sides of length c

3. Two identical rectangles of side lengths b and h

4. Six identical squares with sides of length a

In Exercises 5–7, use the distributive property to determine the surface area of a cube whose edges have the given length.

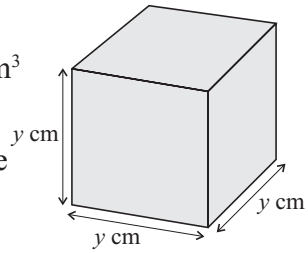
5. Edge length = 4 in **6.** Edge length = 7 cm **7.** Edge length = 11 yd

8. Using the formula, determine the volume of a cube with edges 10 feet long.

In Exercises 9–11, the volume of a cube is given. Determine the length of the edges of each cube.

9. $8\ m^3$ **10.** $27\ yd^3$ **11.** $343\ cm^3$

12. Morgan is designing a water container and he wants it to be big enough for 512 ml water (512 ml water = 512 cm^3 water). He decides to make it in the shape of a cube. What is the suface area of the container he designs?



Lesson 2.3.7 — Expressions and Angles

1. The angles in a triangle always add up to what degree measure?

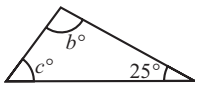
In Exercises 2–3, determine the measure of Angle C inside each triangle.

2. Angle A = 45°, Angle B = 90°, Angle C = ?

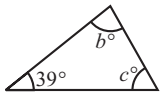
3. Angle A = 30°, Angle B = 70°, Angle C = ?

Use the diagrams in Exercises 4–5 to write equations involving b. Solve them, given that $c = 40$.

4.

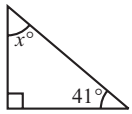


5.

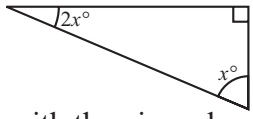


In Exercises 6–8, find the value of x in each diagram.

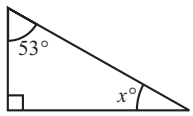
6.



7.



8.



In Exercises 9–11, draw a picture to go with the given description of the angles.

9. The measure of angle MNP is 90°. The measure of angle MNR is $90 - y$.

10. The measure of angle GHJ is x°. The measure of angle GHK is $45 + x$.

11. The measure of angle ABC is 180°. The measure of angle ABD is 90°.

Lesson 2.4.1 — Analyzing Problems

In Exercises 1–3, explain what is happening in each pattern.

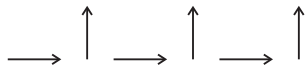
1.



2.



3.



In Exercises 4–9, draw the next shape in the pattern.

4.



5.



6.



7.



8.



9.



In Exercises 10–15, write down the next number in the pattern.

10. 9, 12, 15, 18, 21, 24, …

11. 64, 32, 16, 8, 4, …

12. 5, 15, 25, 35, 45, 55, …

13. 91, 82, 73, 64, 55, 46, …

14. 123, 234, 345, 456, 567, …

15. 96, 84, 72, 60, 48, 36, …

16. Four years ago, Garrison was 6 inches shorter than he is now. At that time, he was 8 inches shorter than his brother. Garrison is now 64 inches tall. How tall was Garrison's brother four years ago?

Lesson 2.4.2 — Important Information

In Exercises 1–4 write down whether you have too much or not enough information to solve each problem. If too much, write down the information you don't need, and solve the problem. If not enough, write down the information you need.

1. Lisa's picture frame has a length of 6 inches and a width of 4 inches. The picture Lisa wants to put in the frame has an area of 30 inches. What is the area, in square inches, of Lisa's picture frame?

2. Each serving in a can of tuna fish contains 60 calories. What is the total number of calories that a can of tuna fish contains?

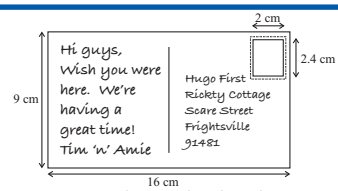
3. Xavier used a flashlight that provides 12 hours of light. If the flashlight were left on for 12 hours, what time would it be when the flashlight no longer provided light?

4. Melanie and Tara have known each other for 12 years. They were both 3 years-old when they met. They each have a birthday on March 18. How old are Melanie and Tara now?

Lesson 2.4.3 — Breaking Up a Problem

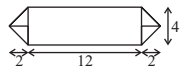
1. Find the last digit of 2^{30} by making a table for small exponents.

2. The diagram on the right shows a postage stamp on a postcard. What area of the postcard is NOT covered by the postage stamp?

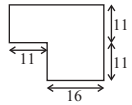


Determine the area of each of the shapes in Exercises 3–5. All measurements are given in inches.

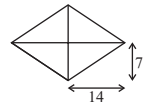
3.

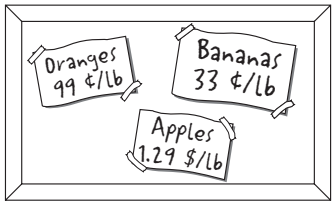


4.



5.





6. When Jeremy saw the signs on the fruit market store window, he went inside to get some fruit.

He bought 3 lb of oranges, 4 lb of apples, and 2 lb of bananas. He had only planned on buying 2 lb of oranges, 2 lb of apples, and 1 lb of bananas. How much additional money did Jeremy end up spending on his fruit?

Lesson 3.1.1 — Understanding Fractions

In Exercises 1–3, write down a fraction that is equivalent to the one that is given.

1. $\frac{6}{18}$ **2.** $\frac{1}{3}$ **3.** $\frac{7}{56}$

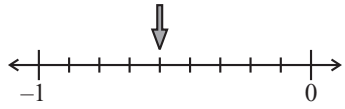
4. Where will a negative fraction with a numerator smaller than the denominator be located on a number line?

In Exercises 5–7, mark each fraction on a number line

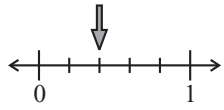
5. $-\frac{3}{5}$ **6.** $-\frac{1}{7}$ **7.** $-\frac{2}{3}$

In Exercises 8–10, write the fraction that is represented on each number line.

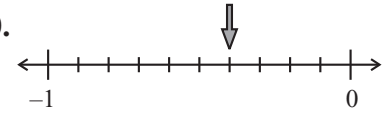
8.



9.



10.

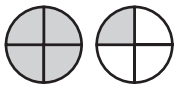


11. Five pairs of Rhonda's 15 pairs of shoes are brown. Express this as a fraction. What fraction of Rhonda's shoes are NOT brown?

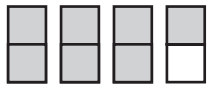
Lesson 3.1.2 — Improper Fractions

In Exercises 1–3, write down the improper fraction and the mixed number each drawing represents.

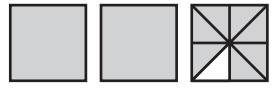
1.



2.



3.



4. Which part of the improper fraction is always the same as in the mixed number?

5. Which 2 operations are used when converting a mixed number to an improper fraction?

In Exercises 6–8, write each mixed number as an improper fraction.

6. $1\frac{8}{9}$ **7.** $8\frac{2}{3}$ **8.** $3\frac{5}{9}$

9. Which operation is used to convert an improper fraction to a mixed number?

In Exercises 10–12, convert the improper fractions into mixed numbers.

10. $\frac{20}{7}$ **11.** $\frac{11}{6}$ **12.** $\frac{23}{3}$

13. Ramon and his friend ate $\frac{3}{2}$ square cakes. Make a drawing to show $\frac{3}{2}$ cakes. Write a mixed number that means the same as $\frac{3}{2}$.

Lesson 3.1.3 — More on Fractions

1. When putting $\frac{9}{8}$ on a number line, what should you do first?

In Exercises 2–4, convert the improper fractions to mixed numbers.

2. $-\frac{18}{7}$ **3.** $-\frac{80}{9}$ **4.** $\frac{36}{8}$

In Exercises 5–7, place the mixed numbers on a number line.

5. $2\frac{5}{7}$ **6.** $-9\frac{2}{3}$ **7.** $-1\frac{1}{2}$

8. Which one of these mixed numbers has the greater value: $-6\frac{3}{4}$, $-6\frac{1}{4}$? Explain your reasoning.

9. Which one of $-\frac{14}{10}$, $-\frac{11}{10}$ has the lesser value? Explain your reasoning.

10. Show these fractions on the same number line. $-\frac{19}{5}$, $-3\frac{2}{5}$

11. On Friday, Jake's thermometer read $-1\frac{1}{3}$ degrees Celsius. On Saturday, Jake's thermometer read $-1\frac{3}{4}$ degrees Celsius. On which day was the temperature greater? Explain your reasoning.

Lesson 3.1.4 — Fractions and Decimals

Convert each of Exercises 1–3 into decimal numbers.

1. $2\frac{3}{15}$ **2.** $8\frac{1}{8}$ **3.** $\frac{13}{12}$

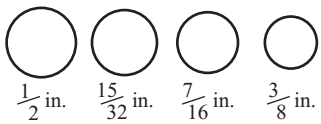
4. Where do a fraction and its equivalent decimal lie on a number line?

In Exercises 5–7, write each fraction as a decimal, without using a calculator.

5. $\frac{3}{4}$ **6.** $\frac{1}{5}$ **7.** $\frac{3}{8}$

8. What does it mean when a decimal is said to be a terminating decimal?

Tristan has four drill bits. The drill bit sizes are shown on the diagram using fractions.



9. Convert each of the drill bit sizes in the diagram to decimals.

10. Tristan needs to drill a hole that is a minimum of $\frac{27}{64}$ inches across. Which of the drill bits could he use?

Lesson 3.2.1 — Multiplying Fractions by Integers

1. Explain how to multiply a fraction by an integer.

2. In the problem $7 \times \frac{2}{3}$ which numbers get multiplied?

In Exercises 3–5, evaluate each expression.

3. $3 \times \frac{3}{4}$ **4.** $11 \times \frac{2}{3}$ **5.** $\frac{6}{8} \times 9$

6. Write down the symbol that goes in the [*space*] in this equation: $\frac{4}{5} \times 16$ [*space*] $16 \times \frac{4}{5}$

In Exercises 7–12, evaluate each expression. Give your answer as an integer or mixed number.

7. $-2 \times \frac{3}{4}$ **8.** $-9 \times \frac{1}{2}$ **9.** $\frac{2}{6} \times -4$

10. $-8 \times 1\frac{1}{3}$ **11.** $7\frac{1}{4} \times -3$ **12.** $5\frac{3}{7} \times 5$

13. Mr. Harris needs to cut pieces of yarn for all of the students in his art class. Each of his 25 students will receive $6\frac{3}{4}$ inches of yarn. Write an expression that can be used to determine the total number of inches of yarn that Mr. Harris will cut. Evaluate your expression. Write your answer as an integer or mixed number AND an equivalent decimal.

Lesson 3.2.2 — More on Multiplying Fractions by Integers

Evaluate the expressions in Exercises 1–3.

1. $3\frac{2}{3} \times 5$

2. $-2 \times 4\frac{1}{4}$

3. $10 \times -5\frac{1}{3}$

4. Three friends are equally sharing 37 feet of fishing line. How many feet will each friend receive?

5. A rectangular table with a perimeter of 22 feet is placed against a wall. $\frac{1}{4}$ of the table's perimeter is touching the wall. What is the total length of the sides of the table that are NOT touching the wall.

Find the error that was made in working Exercises 6–8.

6.
$$2 \times \frac{15}{4} = \frac{15}{8}$$
$$= 1\frac{7}{8}$$

7.
$$-\frac{4}{7} \times 3 = -\frac{12}{7}$$
$$= 1\frac{5}{7}$$

8.
$$3\frac{1}{4} \times 8 = \frac{12}{4} \times 8$$
$$= \frac{96}{4}$$
$$= 24$$

9. Avril's bottle of sunscreen contained 475 milliliters of sunscreen. $\frac{1}{5}$ of the bottle has already been used. Determine the number of milliliters of sunscreen that have already been used. Then, determine the number of milliliters of sunscreen that are left in the bottle.

Lesson 3.2.3 — Multiplying Fractions by Fractions

1. Write down the formula for multiplying fractions using variables *a, b, c, d*.

Evaluate the expressions in Exercises 2–7.

2. $\frac{1}{4} \times \frac{8}{9}$

3. $\frac{29}{10} \times \frac{1}{3}$

4. $1\frac{3}{5} \times \frac{2}{6}$

5. $\frac{9}{16} \times 1\frac{2}{5}$

6. $\frac{2}{7} \times \frac{7}{2}$

7. $1\frac{3}{8} \times \frac{2}{11}$

8. Jenna usually runs for $\frac{3}{4}$ hours. Today she only ran for $\frac{1}{2}$ that time. What fraction of an hour did Jenna run today?

In Exercises 9–11, find the error that was made when the fractions were multiplied.

9. $\frac{1}{3} \times \frac{5}{8} = \frac{5}{11}$

10. $1\frac{1}{4} \times \frac{3}{8} = \frac{3}{32}$

11. $\frac{4}{5} \times \frac{2}{3} = \frac{8}{10}$

12. Becca played $\frac{1}{4}$ of the first $\frac{1}{2}$ of her basketball game. What fraction of the game did Becca play?

13. DeShaun's mom worked $8\frac{5}{6}$ hours on Friday. On Saturday, she worked $\frac{1}{3}$ that many hours. How many hours did DeShaun's mom work on Saturday?

Lesson 3.3.1 — Dividing by Fractions

Use multiplication to solve each of Exercises 1–5.

1. $\frac{1}{4} \div \frac{2}{3}$

2. $\frac{1}{20} \div \frac{2}{5}$

3. $3\frac{1}{3} \div \frac{1}{2}$

4. $\frac{7}{9} \div \frac{5}{6}$

5. $2\frac{3}{4} \div 2\frac{5}{8}$

6. Fill in the four blank spaces in this statement. "If two numbers which are reciprocals of one another are _____, the answer will always be ___. For example, $\frac{4}{5}$ ___ $\frac{5}{4}$ = ___"

7. What is another name for a reciprocal?

In Exercises 8–9 write the following expressions:

8. $m \times \frac{1}{n}$ as a quotient.

9. $\frac{d}{e}$ as a product.

10. Bret has $\frac{1}{2}$ pound of turkey to use for sandwiches. This is shared between 7 equally sized turkey sandwiches. Write an expression to show this situation and evaluate your expression to give the number of pounds of turkey in each sandwich.

Lesson 3.3.2 — Solving Problems by Dividing Fractions

1. What are the 2 types of division problems?

2. $\frac{3}{4}$ of a cup of sugar is shared between 3 people. How much sugar does each person get?

3. An $18\frac{1}{3}$ mile drive is divided into 2 equal-length sections. How long is each section?

4. $50\frac{1}{2}$ oz of juice is divided equally between 4 glasses. How much juice is put into each glass?

5. A 14,000 gallon pool is being drained. The pool drains $2\frac{1}{2}$ gallons of water per minute. How many minutes before the pool is empty?

6. Troy drove from his house to his grandmother's house. After driving 157 miles, he was $\frac{3}{4}$ of the way there. How many miles is it from Troy's house to his grandmother's house?

7. A gardener collected $120\frac{5}{9}$ lb of grass clippings. He took the clippings to a compost site in 3 equally-sized truckloads. What was the weight of grass in each truckload?

8. A tunnel is $14\frac{3}{4}$ feet high. How many sections of 18 in. does this height divide into?

Lesson 3.4.1 — Making Equivalent Fractions

Find the simplest form of each fraction in Exercises 1–3 by canceling.

1. $\frac{8}{12}$ **2.** $\frac{5}{35}$ **3.** $\frac{9}{15}$

4. Equivalent fractions look different but have the same _____ .

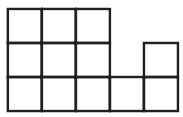
5. Describe what is special about a fraction that's equal to 1. Give an example.

In Exercises 6–8, write down 2 fractions that are equivalent to each given fraction.

6. $\frac{3}{7}$ **7.** $\frac{5}{8}$ **8.** $\frac{9}{12}$

9. Anastasia got $\frac{18}{20}$ of her spelling words correct on her spelling test. What is a simpler form of the fraction of words Anastasia spelled correctly? Write down the decimal equivalent to this fraction.

10. Parker drew a pattern, shown here. Shade $\frac{1}{3}$ of Parker's diagram. Write down 3 fractions that are equivalent to the part you shaded.



Lesson 3.4.2 — Finding the Simplest Form

In Exercises 1–3, find the simplest form of each fraction.

1. $\frac{22}{88}$ **2.** $\frac{5}{13}$ **3.** $\frac{21}{105}$

4. Three-sixteenths of the crayons in Kaia's crayon box are some shade of orange. Write the fraction that means three-sixteenths. Explain whether this is the simplest form of your fraction.

In Exercises 5–7, find the greatest common divisor of each pair of numbers.

5. 45, 30 **6.** 35, 49 **7.** 200, 26

8. Dividing the numerator and denominator of a fraction by their greatest common divisor is the same as doing what?

9. Write the fraction that means the same as 0.16. What is the simplest form of this fraction?

10. Kobi put 50 staples into his stapler when it was empty. He used 15 of the staples for his homework. What fraction of the staples in the stapler did Kobi use for his homework? Write your answer in its simplest form.

Lesson 3.4.3 — Fraction Sums

Find the sum or difference in Exercises 1–6. If possible, simplify your answer.

1. $\frac{9}{12}+\frac{1}{12}$ **2.** $\frac{5}{8}+\frac{2}{8}$ **3.** $\frac{30}{33}-\frac{29}{33}+\frac{10}{33}$

4. $\frac{7}{14}+\frac{8}{14}-\frac{1}{14}$ **5.** $\frac{2}{9}+\frac{2}{9}+\frac{2}{9}$ **6.** $\frac{7}{16}-\frac{2}{16}+\frac{3}{16}$

Three friends shared an 8-slice pizza. Holly ate 1 slice, Malik ate 3 slices, and Darren ate 2 slices. In Exercises 7–11 write down the simplest form of the fraction of pizza that:

7. each friend ate. **8.** Holly and Malik ate. **9.** Holly and Darren ate.

10. Malik and Darren ate. **11.** all 3 friends ate.

12. Using the fraction of pizza that all 3 friends ate, find the fraction of pizza that was left.

Write expressions to represent the following situations, then evaluate your expressions.

13. Luis had a full glass of water. He drank $\frac{3}{4}$ of the water in his glass.

14. Luis then took the water in his glass and added another $\frac{1}{4}$ glass to it.

15. Finally, Luis drank all of the water in his glass.

Lesson 3.4.4 — Fractions with Different Denominators

1. What is a common denominator? Give an example in your answer.

In Exercises 2–7, find a common multiple of each pair of numbers.

2. 3, 5 **3.** 1, 9 **4.** 9, 5

5. 13, 12 **6.** 5, 18 **7.** 12, 7

In Exercises 8–10, work out the sums and differences. Write your answers in their simplest form.

8. $\frac{3}{7}+\frac{2}{3}$ **9.** $\frac{9}{12}-\frac{1}{4}$ **10.** $\frac{2}{13}+\frac{1}{2}$

11. Explain which of these two fractions has a greater value: $\frac{13}{7}$, $\frac{9}{5}$

12. Khalid has grown $1\frac{3}{4}$ inches since last year. Jason has grown $2\frac{1}{3}$ inches since last year. What is the combined growth, in inches, of the two boys since last year? Show your work, and explain each step.

13. In Last Thursday's game, Sherron made $\frac{7}{8}$ of her free throw attempts. Today in practice, she made $\frac{2}{3}$ of her free throw attempts. Explain which is greater, showing what you know about common denominators and comparing fractions.

Lesson 3.4.5 — Least Common Multiples

In Exercises 1–6, find the least common multiple (LCM) of the two numbers.

1. 8, 12 **2.** 4, 12 **3.** 4, 6

4. 22, 20 **5.** 10, 17 **6.** 64, 28

7. What does GCD stand for? Explain how you use it to determine an LCM of two numbers.

8. When adding two fractions, with different denominators, what do you have to do first?

Determine the sums in Exercises 9–11. Write your answers in their simplest form.

9. $\frac{2}{5}+\frac{3}{4}$ **10.** $\frac{4}{17}+\frac{3}{8}$ **11.** $\frac{5}{49}+\frac{3}{2}$

12. Mrs. Greene tried to calculate the sum $\frac{21}{10}+\frac{3}{5}$, giving her answer in the simplest form.

$$\frac{21}{10}+\frac{3}{5}=\frac{42}{20}+\frac{12}{20}=\frac{58}{20}$$

What errors did Mrs. Greene make in working this problem? Correct her errors.

Lesson 3.4.6 — Mixed Numbers and Word Questions

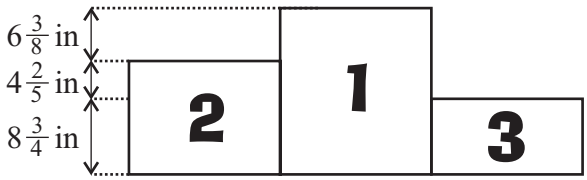
1. What do you do if the fraction parts of the mixed numbers you're adding have different denominators?

2. When fractions have the same denominators, how can you compare them?

In Exercises 3–5, determine the sum or difference.

3. $5\frac{1}{2}+8\frac{2}{3}$ **4.** $12-7\frac{5}{9}$ **5.** $19\frac{1}{2}+12\frac{3}{5}$

In Exercises 6–8, use this diagram of a podium. Give all your answers as simplified fractions.



6. Kyle took 2nd place in a race. How far did he have to climb to get onto his position on the podium?

7. Julio came 1st. How far did he have to climb?

8. Lin came 3rd. What was the difference in height between Lin's and Julio's podium positions.

Lesson 3.5.1 — Fractions and Percents

1. Explain how to use equivalent fractions to convert $\frac{3}{25}$ into a percent.

In Exercises 2–7, write each fraction as a percent.

2. $\frac{19}{50}$ **3.** $\frac{14}{35}$ **4.** $\frac{50}{2500}$

5. $\frac{363}{500}$ **6.** $\frac{13}{40}$ **7.** $\frac{3}{14}$

In Exercises 8–10, write each percent as a fraction in its simplest form

8. 40% **9.** 65% **10.** 22.5%

11. Archie is watching birds for a project. $\frac{64}{144}$ of these birds are Northern Mockingbirds. What is this as a percentage?

12. A group of people are kayaking down a river. 37.5% of the group capsized. Write the fraction of the group that capsized in its simplest form.

13. A soccer team has 11 players on the field. One of the players is sent off. Write the fraction and the percentage of the original team that remains on the soccer field.

Lesson 3.5.2 — Percents and Decimals

1. Explain how to write any percent as a decimal.

In Exercises 2–7, the values are shown as a percent, a decimal, or a fraction. Write each value in the two forms which are not shown. Give all fractions in their simplest form.

2. 37% **3.** 0.64 **4.** 0.05

5. $\frac{8}{42}$ **6.** 99.6% **7.** 33.8%

In Exercises 8–10, order each set of numbers from least to greatest.

8. $\frac{2}{3}$, 0.637, 65.2% **9.** 0.102, $\frac{1}{8}$, $\frac{10}{85}$ **10.** $\frac{3}{55}$, 0.006, 5%, $\frac{4}{70}$

11. A football team has 46 active players. Only 11 of them are permitted on the field at any one time. What percentage of the players in this team must remain off the pitch?

12. Tommy does a survey of his class on the colour of pens they use. 18 use black pens, 9 use blue pens, 4 use green pens and 1 uses a red pen. What percent of the class use blue pens?

Lesson 3.5.3 — Percents of Numbers

In Exercises 1–3, find the following percents:

1. 33% of 55 **2.** 12% of 60 **3.** 95% of 108

4. 69% of 69 **5.** 13% of 89 **6.** 15% of 564

7. It is thought that about 42% of people in California speak languages other than English at home. How many people from a class of 31 would you expect to speak a non-English language at home?

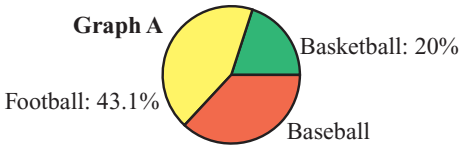
8. Monica takes a 500 ml bottle of water with her on her morning run. When she returns she has 24% of the water remaining. How many ml of water are left in the bottle?

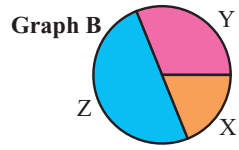
9. Monica's friend joins her on her run the next day, and returns with 142 ml water in her bottle. This is 25% of the amount she started out with. How much did she have to begin with?

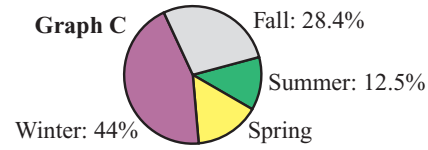
10. Yoshiko did a survey and found that 56% of people said their favorite color was red, and 24% said that their favorite color was blue. If 18 people opted for blue, how many chose red?

Lesson 3.5.4 — Circle Graphs and Percents

Exercises 1–5 use the following circle graphs.







1. What are the missing percents in circle graphs A and C?

2. Graph A refers to a group of 65 people who were asked which sport they preferred, from a choice between football, basketball and baseball. How many people said basketball?

3. Graph B gives the results of an environmental traffic survey, detailing the number of people traveling in each car. A total of 492 cars were observed and the following results obtained: 1 person only: 246 cars, 2 people: 153 cars, 3+ people: 93 cars. Which section of the graph represents each group, and what percentage of the total do they each make up?

4. Graph C shows the annual sales made by a shop in each of the four seasons. If the sales made in summer came to a total of $45,843, how much money was taken during the winter season?

5. The following year, the shop took an annual total of $500,000, but the percentage of the takings in each season remained the same. How much more was made in the fall season of this year compared to the previous one?

Lesson 3.5.5 — Percent Increase

1. If you multiplied a number by 1.09, what percentage increase would you be calculating?

In Exercises 2–4, calculate the total after the increase:

2. Increase 64 by 25% **3.** Increase 140 by 8% **4.** Increase 87 by 65%

5. Explain the steps you would take to calculate a 151% increase to a number.

In Exercises 6–7, calculate the total after the increase:

6. Increase 35 to 134% of its original value **7.** Increase 65 to 245% of its original value.

8. Vincent's bill at a restaurant is $45, he leaves a 16% tip. How much did he pay in total?

9. DeMorris's taxi fare came to $13.40, but he rounded up to $15 to include the tip. What percentage tip did DeMorris add to the fare?

10. Sharon pays $65.27 for a meal, including a 22% tip. How much did her meal cost before the tip?

Lesson 3.5.6 — Percent Decrease

In Exercises 1–3, calculate the total after the increase:

1. Decrease 64 by 25%

2. Decrease 140 by 8%

3. Decrease 87 by 65%

4. If you multiplied a number by 0.8, what percentage decrease would you be calculating?

In Exercises 5–7, work out the percentage decrease from the first number to the second.

5. 178 decreased to 78

6. 18 decreased to 2

7. 579 decreased to 96.5

8. Alison works at a kayak shop and can buy kayaks for trade price. Trade price is 57% of the retail price. If she buys a kayak that should retail at $1049, how much will she actually pay?

9. Three t-shirts are in a sale. Their original price, and the discount are shown in the following table. Write the t-shirts in ascending order of price following the discount.

T-Shirt #	1	2	3
Price	$16	$21	$28
Discount	5%	25%	45%

10. Arnold buys a laptop for $599 after negotiating a discount of 26%. How much would it have cost before the discount?

Lesson 3.5.7 — Simple Interest

Enoc borrows $800. He makes a $945 payment after 36 months which pays off his loan.

1. What was the Principal?

2. How much interest did he have to pay?

3. How many years did it take Enoc to repay his loan?

4. The formula for simple interest is $I=Prt$. What do each of the variables mean, and what are the most common units of I, P and t?

Use the formula for simple interest from Exercise 4 to answer Exercises 5–8.

5. Simone borrowed $8500. She paid back the interest 18 months later. If the rate she paid interest at was 11.5% per year, how much interest did she have to pay?

6. Elvera puts $1655 into a savings account which pays 8% interest. 48 months later she checks her balance. How much should she have?

7. Benny has $1651 in his bank account. Three years ago he paid in $1300. What is the rate of interest on his account?

8. Juana pays off her loan with a single payment of $998.75. $148.75 of this is interest. If the interest rate was 7%, how long has she had this loan for?

Lesson 4.1.1 — Ratios

Chloe has a bag of marbles. The bag contains 7 green marbles, 2 blue marbles, 5 red marbles, 1 orange marble and 4 purple marbles. Write down the ratios of the following colors of marbles:

1. Green to orange

2. Red to purple

3. Orange to blue

4. Purple to green

5. Blue to red

In Exercises 6–8, write the ratio 2 other ways than the way it is given.

6. 6 to 2

7. 2:1

8. $\frac{5}{7}$

9. There are 4 volleyball players and 3 soccer players in Tim's class. What is the ratio of volleyball players to soccer players? Write your answer three different ways.

10. The ratio of squirrels to acorns is 3:1. Express the ratio of acorns to squirrels in three different ways.

Felix wrote the following ratio to compare the length of 2 pencils: 8 inches: 6 inches

11. What mistake did Felix make with his ratio?

12. Write the correct ratio of longer to shorter pencil length in two different ways.

Lesson 4.1.2 — Equivalent Ratios

In Exercises 1–6, write down an equivalent ratio that is simpler than the given ratio.

1. 5:15 **2.** $\frac{10}{100}$ **3.** $\frac{18}{6}$

4. 90 to 15 **5.** 13:39 **6.** 16 to 180

Carlito bought 4 red apples, 8 yellow apples, and 2 green apples at the store.

7. What is the ratio of yellow apples to green apples? Write your answer in its simplest form.

8. What is the ratio of red apples to yellow apples? Write your answer in its simplest form.

9. What is the ratio of red apples to green apples?

Abe has 20 CDs. There are 10 pop CDs, 2 rock CDs, 3 rap CDs, and the rest are country CDs. Write the following ratios in their simplest form:

10. pop CDs to rap CDs **11.** rock CDs to pop CDs

12. rap CDs to country CDs **13.** country CDs to pop CDs

14. Find a number x, so that the ratio 2:7 is equivalent to x :35.

15. Find a number y, so that the ratio 3:13 is equivalent to y :39.

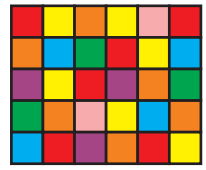
Lesson 4.1.3 — Proportions

In Exercises 1–4, write down whether the ratios can be written as proportions. Explain your answer.

1. 4:3 and 8:3 **2.** 2:5 and 4:10

3. 1:4 and 2:6 **4.** 6:1 and 18:3

Shawna is tiling the area around a sink, using 6 tiles like the one shown on the right. Write a proportion involving each of the following:



5. The numbers of red squares and purple squares in 1 tile and 6 tiles.

6. The numbers of green squares and pink squares in 1 tile and 6 tiles.

7. The numbers of blue squares and orange squares in 1 tile and 6 tiles.

A fruit cup was made using 6 parts strawberries and 5 parts raspberries.

8. Write a proportion involving the number of strawberries and raspberries needed in 1 fruit cup and 8 fruit cups.

9. If s is the number of strawberries needed to mix with 20 raspberries, solve for s.

10. If r is the number of raspberries needed to mix with 36 strawberries, solve for r.

Lesson 4.1.4 — Proportions and Cross-Multiplication

1. Explain what it means to cross multiply.

In Exercises 2–4, use cross-multiplication to check if the ratios are equivalent.

2. $\frac{2}{5}$ and $\frac{14}{35}$ **3.** 3:2 and 21:12 **4.** 6 to 5 and 42 to 35

In Exercises 5–7, determine the value of x in each proportion.

5. $\frac{7}{2} = \frac{14}{x}$ **6.** $\frac{x}{5} = \frac{36}{10}$ **7.** $\frac{42}{x} = \frac{6}{7}$

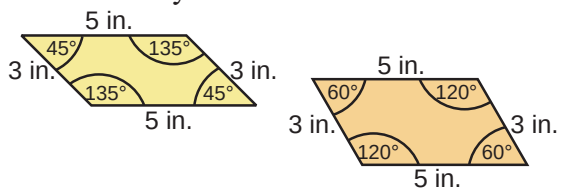
8. Paul is packing for a trip to a basketball tournament. For each day of the tournament, he needs 2 pairs of socks. Is he correct to pack 8 pairs of socks for a 3-day tournament? Explain your answer. Include a proportion.

9. A baker has 20 cups of flour and 12 cups of sugar to use for muffins. One batch of muffins contains 2 cups of flour and $\frac{1}{2}$ cup of sugar. How many batches of muffins can the baker make? Will any flour or sugar be left over? Explain your reasoning.

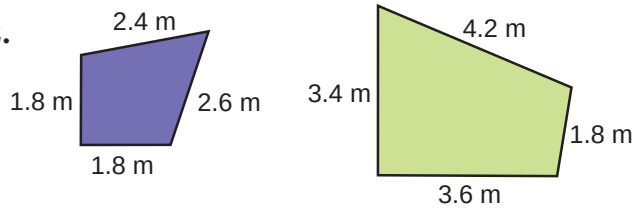
Lesson 4.2.1 — Similarity

Exercises 1–2 each show a pair of shapes. Explain whether the shapes in each pair are similar. Give reasons for your answers.

1.

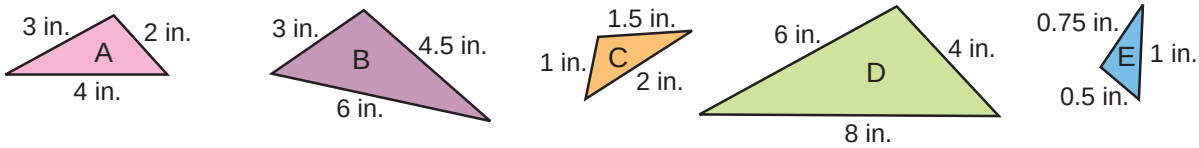


2.



3. Explain what ratios have to do with shapes being similar.

The triangles below are all similar to each other. Use the diagrams to answer Exercises 4–7.



Find the ratios of the length of corresponding sides between each of the following pairs of triangles:

4. A and C

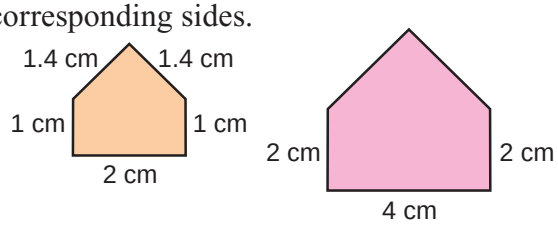
5. D and B

6. E and A

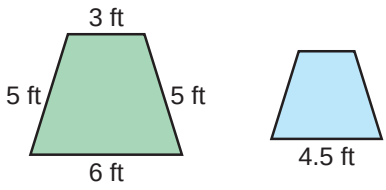
7. B and C

In Exercises 8–9, find the missing lengths in each pair of shapes by determining the ratios of the lengths of corresponding sides.

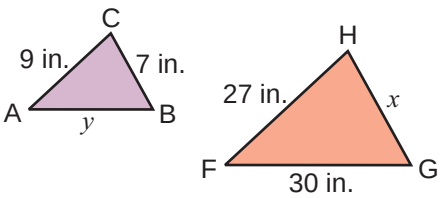
8.



9.



Lesson 4.2.2 — Proportions in Geometry



Use the diagram on the left to answer Exercises 1–4.

1. Write a proportion using the ratios AC:FH and BC:GH.

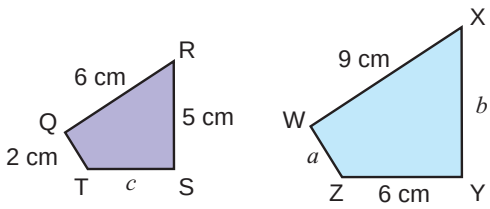
2. Solve your proportion to find x.

3. Write a proportion using the ratios AC:FH and AB:FG.

4. Solve your proportion to find y.

5. If you know the ratio of the lengths of 2 sides in one shape, what can you say about the ratio of the corresponding sides in any similar shape?

6. The two figures shown on the right are similar. Use proportions to find the missing lengths a, b and c.



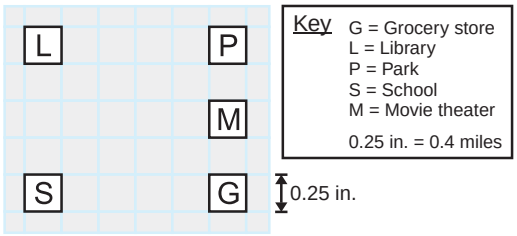
7. Carlita has two photos on her wall. One photo measures 5 cm by 8 cm, and the other measures 6 cm by 9 cm. Are the shapes of the two photos similar? Use a proportion in your explanation.

8. Marcus has cut a figure with sides that measure 11 inches, 15 inches, and 17 inches in length. He wants to cut a second figure that is similar to his first figure. The length of the longest side of the new figure is 51 inches. Write and solve a proportion to find the lengths of the other two sides Marcus should cut.

Lesson 4.2.3 — Scale Drawings

The map below represents the location of some buildings in a small town.
The scale is 0.25 inches = 0.4 miles, and the grid squares on the drawing measure 0.25 in. by 0.25 in.
In Exercises 1–6, calculate each of the following distances in miles.

1. From the library to the school.
2. From the park to the library.
3. From the movie theatre to the park.
4. From the grocery store to the movie theatre.
5. From the park to the grocery store.
6. From the school to the grocery store.



Donovin has a drawing of a pencil made with a scale of 1 mm : 0.025 inches.
7. The length of the pencil in the drawing is 240 millimeters. What is the length of the real pencil?
8. The real pencil is 0.25 inches wide. How wide is the pencil in the drawing?
9. The real pencil's eraser is 0.4 inches long. How long is the eraser in the drawing?

A builder made a scale drawing of a retail store using a scale, in inches, of 2:y.
The height of the store in the drawing is 3 inches.
10. The height of the actual store is 12 feet. Find the value of y.
11. The width of the actual store is 50 feet. Find the width of the store in the drawing.

Lesson 4.3.1 — Customary and Metric Units

The length of a painter's ladder is 8 yards.
1. What is the length of the ladder in feet?
2. What is the length of the ladder in inches?

3. When you convert to a smaller unit, the number of units gets _____ .
4. When you convert to a larger unit, the number of units gets _____ .

Convert the values in Exercises 5–9.
5. 12 miles to yards
6. 4 feet to inches
7. 144 inches to feet
8. 8800 yards to miles
9. 2 miles to inches
10. 4500 inches to yards

The height of a tree in Erin's garden is 2.25 meters.
Find the height of the tree in each of the following units:
11. decimeters
12. centimeters
13. millimeters
14. hectometers

Lesson 4.3.2 — Conversions and Proportions

1. What is the ratio of inches to miles? Explain your answer.

2. Convert 342.25 mm to meters. Use a proportion to determine your answer.

Keyshawn ran a 1600 meter race. Use proportions to convert the race distance to each of the following:
3. dekameters
4. hectometers
5. kilometers

6. What is the ratio of inches to yards?
7. Convert 9.37 yards into inches using your ratio from Exercise 7.

A professional baseball player hit a home run ball 527 feet. Use proportions to calculate the following:
8. the number of inches the ball was hit
9. the number of yards the ball was hit
10. the number of miles the ball was hit

Lesson 4.3.3 — Converting Between Unit Systems

A flight from Minnesota to Mexico took 6.25 hours. Convert the length of the flight into each of the following units. Use a proportion for each conversion.

1. minutes **2.** seconds **3.** days

The largest anaconda in the city zoo is 28 feet long.

4. Use a proportion to convert 28 ft to inches.

5. Use a proportion to convert 28 ft to yards.

6. Use the conversion factor 2.54 to convert 28 ft to centimeters.

7. Use a proportion to convert 28 ft to meters.

Darla and Rex converted 1 foot into meters. Their work is shown below.

Darla's Conversion

feet to inches:	1 foot	= 12 inches
inches to cm:	12 × 2.54	= 30.48 cm
cm to m:	30.48 ÷ 100	= 0.3048 m

Rex's Conversion

1 yard = 0.91 meters

1 foot = $\frac{1}{3}$ yard

0.91 ÷ 3 = 0.3033 m

8. Explain how Darla did her conversion. Is her conversion correct?

9. Explain how Rex did his conversion. Is his conversion correct?

10. Why are Darla's and Rex's answers close but not equal?

Lesson 4.3.4 — Other Conversions

In Exercises 1–3, say which of the temperatures given is higher.

1. 50 °F or 8 °C **2.** 5 °F or –10 °C **3.** 170 °F or 79 °C

The temperature at 7 a.m. was 12 °C. By 10 a.m. the temperature had increased by 21 °F.

4. What was the temperature at 10 a.m. in °F?

5. What was the temperature at 10 a.m. in °C?

Scientists often measure temperature in units called Kelvins (K), instead of °F or °C.
If K is a temperature in Kelvin, then: K = C + 273.15
Convert the following:

6. 40 °C to K **7.** –116 °C to K **8.** 86 °F to K

9. 200.15 K to °C **10.** 453.2 K to °C **11.** 68 K to °F

Lesson 4.4.1 — Rates

In Exercises 1–4, use the given quantities to work out two unit rates.

1. 30 players for 15 basketballs **2.** $2.40 for 12 cookies

3. 12 mouse toys for 3 cats **4.** 288 dominoes shared between 24 containers

5. Two chefs each bought salmon for their restaurant. Chef Ana got 12 pounds of salmon for $83.88. Chef Todd got 15 pounds of salmon for $95.85. Which Chef got a better price per pound of salmon? Explain your answer.

6. A van owner and a SUV owner were comparing the efficiency of their vehicles. The van drove 108 miles on 9 gallons of gas. The SUV drove 240 miles on 16 gallons of gas. Which vehicle drove more miles per gallon of gas? Show your work and explain your answer.

A small gym has 40 45-pound weights, 75 35-pound weights, and 90 25-pound weights that weightlifters can use. There are 5 racks that each hold an equal number of each type of weight.

7. How many of each type of weight are there per rack?

8. What is the total weight in pounds held by each rack?

Lesson 4.4.2 — Using Rates

1. Dora made 9 necklaces using 36 beads per necklace. How many beads did Dora use in total?

2. Joey's mom has 6 strings of outdoor lights with 250 lights per string. Find the total number of lights.

3. Judy ran 10.5 miles at a speed of 6 miles/hour. How long, in minutes, was Judy running for?

4. Aaron spent \$47.76 on mulch for his yard. If the mulch costs \$1.99/yd^3, how many yd^3 of mulch did Aaron buy?

Lyron and Gabriela are each making a car journey.

5. Lyron's car travels 140 miles in 240 minutes. What is his average speed in miles per hour?

6. Gabriela's average speed is 40% faster than Lyron's. What is Gabriela's average speed?

7. How long would it take Gabriela to travel 180 miles at this speed?

8. How far would Gabriela travel in 90 minutes at this speed?

Lesson 4.4.3 — Finding Speed

1. Victoria rides her bike for half an hour. If she travels a total of 7 miles, what is her average speed?

2. Andres walks a distance of 10 miles in 3 hours. What is his average speed?

3. An airplane cruises at a constant speed for a distance of 1200 miles.
What is the speed of the airplane if it takes 2.25 hours to travel this distance?

An Indy car once recorded a speed of 239 miles/hour.

4. What is this speed in miles/minute?

5. What is this speed in miles/second?

6. What is this speed in kilometers/hour? (Use the conversion factor 1 mile = 1.61 km)

7. What is this speed in meters/second?

8. A dragonfly flies at a speed of 20 miles per hour.
A hornet flies at a speed of 21.5 kilometers per hour.
A bee flies at a speed of 4 meters per second. Convert the speeds of the bee and dragonfly to km/hour.
Which of these insects flies at the fastest speed? (Use the conversion factor 1 mile = 1.61 km)

Lesson 4.4.4 — Finding Time and Distance

1. How can you find the distance an object travels in a certain amount of time if you know its speed?

Malik walks at an average speed of 0.6 meters/second.
Jessica walks at an average speed of 3.5 miles/hour.

2. If Malik walks for 9000 seconds, how many meters will he travel?

3. If Jessica walks for 2.5 hours, how many miles will she travel?

4. Which person walked for a longer period of time? Explain your reasoning.

5. Which person walked a greater distance? Explain your reasoning.

6. The data below shows the speed that two toy robots moved at, and the time they traveled for.
Explain how you could compare the two distances that could be calculated from the data.
You don't need to work out the actual distances traveled.

56 cm/second for 5 minutes

40 ft/hour for 2 hours

Suri set a treadmill to move at a speed of 6.5 miles/hour. She ran on the treadmill at that rate for 45 minutes and 30 seconds.

7. What is the distance Suri ran in miles?

8. Calculate Suri's speed in meters/second?

9. What is the distance that Suri covered in kilometers?

Lesson 4.4.5 — Average Rates

David went on a bike ride. He cycled at an average speed of 29 miles/hour for the first hour. During the next 2 hours, he biked at an average speed of 21 miles/hour.

1. How many miles did David cycle during the first hour?
2. How many miles did he ride during the next two hours?
3. What is the total distance David rode?
4. What was his average speed for the three hours?

Carmen ran at a rate of 5.5 miles/hour for 0.75 hours. During the next 0.5 hours, she ran at a rate of 6.5 miles/hour.

5. How far did Carmen run in the first 0.75 hours?
6. How far did Carmen run in the next 0.5 hours?
7. What was the total distance that Carmen ran?
8. What was the total length of time she ran for?
9. What was Carmen's average speed in miles/hour?
10. What was Carmen's average speed in meters/second?

Lesson 5.1.1 — Median and Mode

In Exercises 1–2, find the median of each data set.

1. {12, 10, 15, 14, 17, 11, 20}
2. {25, 53, 12, 46, 31, 45, 72, 11}

In Exercises 3–4, find the mode of each data set.

3. {27, 21, 27, 23, 29, 25, 27, 21}
4. {2, 15, 11, 2, 16, 5, 2, 2, 12}

5. Can a data set have 2 modes? Explain your answer.
6. Can a data set have no mode? Explain your answer.

Item	Price
Soup	$1.25
Salad	$2.00
Sandwich	$1.75
Hot dog	$1.50
Tacos	$1.50

Use the lunch menu shown on the left for Exercises 7–10.

7. What is the median price on the lunch menu?
8. What is the mode of the prices on the menu?
9. Two other items, "Hamburger and Fries" and "Grilled Cheese and Fries" will be added to the sign tomorrow, each with a price of $2.25. What will be the new median price on the menu?
10. What will be the new mode price on the sign?

Lesson 5.1.2 — Mean and Range

In Exercises 1–3, calculate the mean of the data in each set.

1. {6, 1, 3, 2, 9, 14, 11, 20}
2. {14, 14, 11, 9, 4, 7, 13}
3. {60, 12, 15, 57}

4. In his first 5 games, Amare averaged 10.8 rebounds/game. He had the rebounds listed in the data set {9, 12, 7, 15}, but he was missing one of the rebound numbers. Use the mean to determine Amare's missing rebound number.

5. The mean of 5 numbers is 1. Explain why the set of data cannot be {0, 2, 4, 3, 3}.

6. Explain why range is not considered a measure of central tendency.

Below is a list of the prices of eight shirts on a clothing store rack:
$24.99, $16.99, $15.99, $18.99, $9.99, $8.99, $15.99, $19.99

7. What is the range of shirt prices on the rack?
8. What is the mean shirt price on the rack?
9. What is the median shirt price on the rack?
10. What is the mode shirt price on the rack?

11. Write a data set that meets the following criteria:

Mode = 10	Mean = 10.6	Median = 10
Range = 20	Greatest value is 22	
5 pieces of data	Mode occurs twice in data set	

Lesson 5.1.3 — Extreme Values

1. What is an outlier?
2. What can outliers do to the mean of a data set?

The data below shows the number of days 7 neighborhood houses were up for sale before they sold.
{7, 4, 9, 6, 92, 5, 10}
3. Identify the outlier.
4. Calculate the mean with the outlier.
5. Calculate the mean without the outlier.
6. Explain which mean would be better to use as a typical value.

The number of pairs of glasses prescribed by an eye doctor each day for 6 days is shown below.
{20, 6, 4, 3, 4, 5}
7. Find the mean.
8. Explain why the mean of this data set is not a good choice as a typical value.
9. Find the mode.
10. Find the median.
11. What is the most representative measure of central tendency for this data set?

Lesson 5.1.4 — Comparing Data Sets

Give a summary of each set of data in Exercises 1–3.
1. {2, 18, 15, 8, 15, 20}
2. {14, 8, 8, 8, 4, 15}
3. {6, 17, 1, 17, 3, 4}

The statements in Exercises 4–5 are about the three data sets in Exercise 1–3.
Say whether you agree with each statement, and explain your answers.
4. The spread of the data is greatest in Exercise 3.
5. The data set in Exercise 1 has the highest typical value.

Two gymnasts' scores for their 4 events at a gymnastics competition are listed below.

Katie	{8.60, 9.75, 9.15, 9.90}
Jazmin	{9.85, 9.00, 8.55, 9.20}

6. Find the mean, median, mode, and range of scores for each gymnast.
7. Use your answer to Exercise 5 to compare the sets of scores for the two gymnasts.
8. What does it mean if your data set has a small range?
9. What could it mean if your data set has a very large range?

Lesson 5.2.1 — Including Additional Data: Mode, Median and Range

1. Is there a value that could be added to the data set {5, 3, 1, 2, 2, 3, 4, 7} to change the mode? Explain your answer.

2. How could you add as many values as you want to a data set without changing the range?

The data set below contains the number of brothers and sisters of each student in Terrell's class.
{0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 4, 5, 6, 6, 8}
3. What is the mode of this data set?
4. How would the mode be affected if another student with 1 brother were added to the data set?
5. What is the median of this data set?
6. How would the median change if another student with 2 sisters and 1 brother was added?
7. What is the range of this data set?
8. What value could you add to the data set to change the range to 13? Explain your answer.
9. Terrell says that there is no value you could add to the data set to change the range to 5. Is he correct? Explain your answer.

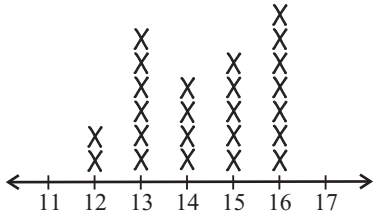
Lesson 5.2.2 — Including Additional Data: Mean

1. What kind of effect does adding values near the original mean have on the mean?
2. What kind of effect can adding an outlier to the data set have on the mean?

The grade point averages of the starting players on Lily's basketball team are: {3.25, 3.25, 3.5, 3.75, 4.0}
The grade point averages of the 7 reserve players on the team are: {3.0, 3.4, 3.5, 3.67, 3.8, 3.9, 4.0}
In Exercises 3–5, find the mean grade point average of the following:
3. The 5 starting players. **4.** The 7 reserve players. **5.** All 12 players the team?
6. A new reserve player joins the team with a 3.15 grade point average. How does this change the mean grade point average of the reserve players?
7. How does this change the mean grade point average of all players?
8. Was the change you found in Exercise 7 large or small? Explain why adding a value of 3.15 affected the mean in this way.

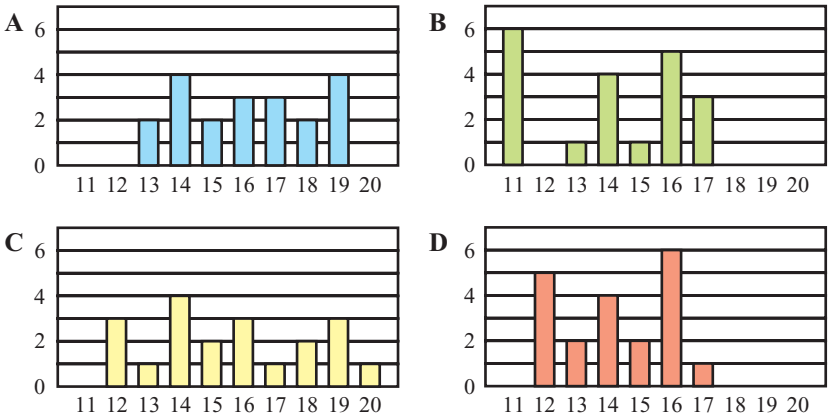
9. Jordan has been told that the mean of a data set is 15 and the mean of a second data set is 17. He wants to find the mean of the two data sets combined. He decides the mean of the combined set must be (15 + 17) ÷ 2 = 32 ÷ 2 = 16. Is Jordan's method correct? Explain your answer.

Lesson 5.3.1 — Analyzing Graphs



Use the data display on the left to answer Exercises 1–3.
1. What type of data display is this?
2. What is the mode of the data?
3. What is the range of the data?

Use the bar graphs below to answer Exercises 4–8.
4. Which data set has the greatest range?
5. Which data set has a mode of 16?
6. Which data set has more than one mode?
7. Which two data sets have the same range?
8. Which data set has the smallest mode?



Miss Allaire's science class went on a nature walk to collect leaves.
The number of leaves each student in the class collected is displayed below.
9. Who collected the most leaves? How many did he/she collect?
10. Who collected the least number of leaves?
How many did he/she collect?
11. Which student collected exactly 7 leaves?
12. What was the total number of leaves collected?
13. What is the mode number of leaves collected?
14. What is the range in number of leaves collected?

Student	Number of leaves collected
Amir	
Cecilia	
Chad	
Mark	
Tawana	

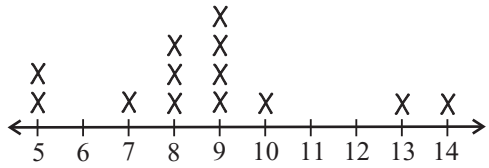
Key: = 2 leaves

Lesson 5.3.2 — Finding the Mean and Median from Graphs

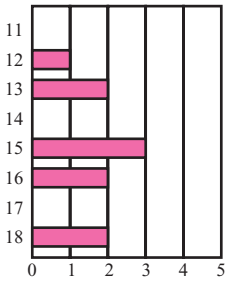
Use the line plot on the right for Exercises 1–2.

1. Find the median of this data set.

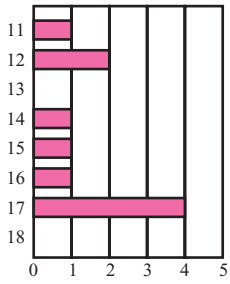
2. Determine the mean of this data set.



Graph A



Graph B



Use the the bar graphs on the left to answer Exercises 3–6.

3. Which graph has the greatest mode?

4. Which graph has the greatest range?

5. Which graph has the greatest median?

6. Which graph has the greatest mean?

Hayley collected data from her classmates as to the number of people each student had in their family. She used the data to make the bar graph below.



7. What is the mean number of people in the students' families?

8. What is the median number of people in the students' families?

9. What is the range in number of people in the students' families?

10. Find the new mean and median if the value 15 is added to Hayley's data set.

Lesson 5.3.3 — Other Types of Graph

Mrs. Martinez asked the girls and boys in her class to choose their favorite from four colors. She made the circle graphs below to display the girls' data and the boys' data.

1. Which color was most popular among the girls?

2. Which color was most popular among the boys?

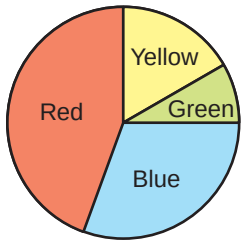
3. Which color was least popular among the girls?

4. Which color was least popular among the boys?

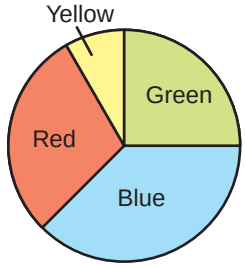
5. Which group of students liked yellow more?

6. Which group of students liked green more?

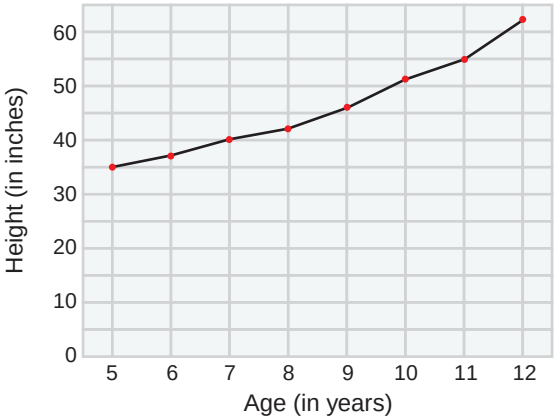
7. Mrs. Martinez asked the class how many girls and boys voted for red. Kirk said 11. Explain the problem with Kirk's answer.



Girls



Boys



Abby's mom made the data display on the left to show Abby's height between the ages of 5 and 12.

8. Identify this type of data display.

9. What was Abby's height at age 5?

10. How old was Abby when her height was exactly 55 in.?

11. How much did Abby grow between the ages 7 of 11?

12. Between which two ages did Abby grow the most?

13. Explain the difference between a single bar graph and a double bar graph.

Lesson 5.4.1 — Using Samples

3000 girls were surveyed for a study to determine the number of hours of television teens watch per day.

1. Explain why this study is biased.

2. How could this sample be changed to be more representative of the population?

Identify the population and the sample in Exercises 3–5.

3. A uniform company wants to know what style of jerseys college basketball players prefer. The company sent jersey samples to 500 of their college clients to find out which the players liked best.

4. A pool manager needs to determine the amount of chlorine in the adult pool versus the amount of chlorine in the children's pool. A small bottle of water is filled from each of the pools.

5. A veterinary office wants to know whether to add dog grooming services to the list of services they offer. A receptionist calls 50 dog owners who use the vet office to ask if they'd be interested.

6. Explain why a sample of 10 people out of a city of 100,000 won't give very useful results.

7. A meeting was held on Tuesday at 10:00 a.m. to discuss a new school policy on the use of cell phones. People at the meeting were surveyed as to whether they thought cell phones should be allowed on school grounds. Based on these results, a school policy was made banning the use of cell phones on school grounds. Give 3 reasons why this sample may have been biased.

Lesson 5.4.2 — Convenience, Random and Systematic Sampling

In Exercises 1–3, identify the type of sampling being used, and give an advantage and a disadvantage of using that type of sampling in the situation described.

1. A mall manager is surveying mall shoppers as to what kind of store they would like to see open in the vacant store area. She interviewed 50 shoppers who walked by her mall office.

2. A TV news station asks random people coming out of a movie theatre grand opening what they think about the theatre.

3. Every 3rd person coming to a bank drive thru between 1–2 p.m. was surveyed as to whether bank hours should be extended.

A cell phone store wants to know if customers are pleased with the service plans that are available to them through their company.

4. Give an example of convenience sampling that could be used in this situation.

5. Give an example of random sampling that could be used in this situation.

6. Give an example of systematic sampling that could be used in this situation.

Lesson 5.4.3 — Samples and Accuracy

1. If a sample is representative, what can you say about how it compares to the population?

2. If a sample accurately represents a population, what can you say about the sampling error?

The mass of 450 different laboratory mice was determined and recorded.
The data below shows the mass in grams of 4 different samples of mice in the laboratory.

Sample A: {4.5, 3, 3.2, 5, 2.9}
Sample B: {6.1, 4.4, 3, 2.8, 4.6}
Sample C: {6.5, 3.1, 4, 3.7, 5}
Sample D: {5.9, 3.1, 2.6, 3.8, 4.2}

3. Find the mean of each of the 4 samples.

4. Find the median of each of the 4 samples.

The mean mass of the population of mice is 5.75 grams.

5. Find the sampling error for Sample A.

6. Find the sampling error for Sample B.

7. Find the sampling error for Sample C.

8. Find the sampling error for Sample D.

9. The sampling error for the median between Sample A and the population is 1.25 grams . What is the median of the population?

10. Based on your median in Exercise 9, find the sampling error for the median for Samples B, C, and D.

Lesson 5.4.4 — Questionnaire Surveys

A company wants to collect data on what types of junk foods people in Texas buy at the grocery store. The store surveys shoppers as they are leaving the store, asking about their junk food purchases.

1. What is the population?

2. What is the sample?

3. What are the benefits of collecting data this way?

4. What are the problems with collecting data this way?

5. What is a biased question?

6. When might a biased question be asked?

A student council is trying to gather information about whether a school should change its school colors.

7. Write an unbiased question that could be used to gather opinions in the survey.

8. Write a biased question in favor of changing the school colors.

9. Write a biased question against changing the school colors.

200 sixth graders were asked, "Isn't it unfair that teachers aren't graded on their teaching?" As a result a claim was made that 100% of students think that they should get an opportunity to evaluate their teachers' performance.

10. Is this a valid claim? Explain your reasoning.

11. Write an unbiased question that could have been used to gather more valid opinions.

Lesson 5.5.1 — Evaluating Claims

Ms. Clark made the line plot on the right to show her students' scores on their last spelling test. She showed the line plot to the class and they discussed their results.

In Exercises 1–6, explain whether the data from the line plot supports each claim.

1. Martin said, "Most of us scored an 18 on our last test."

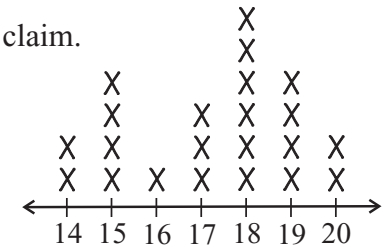
2. Trey said, "The mode score on the test was 14 and 20."

3. Liz said, "At least this time the range of scores was only 6."

4. Attiana said, "The median score was 17 on this test."

5. Write a different claim that is supported by the data on the line plot.

6. Write a different claim that is not supported by the data on the line plot.



Spelling test scores

A teacher made the chart below to show the first 3 test scores of 3 different students in her class.

Student	Test 1	Test 2	Test 3
Angelo	89	95	92
Brianna	94	81	92
Corinne	85	89	94

7. Write a claim about Angelo's test scores that can be supported by the data.

8. Write a claim about Angelo's test scores that cannot be supported by the data.

9. Write a claim about Brianna's test scores that can be supported by the data.

10. Write a claim about Brianna's test scores that cannot be supported by the data.

11. Write a claim about Corinne's test scores that can be supported by the data.

12. Write a claim about Corinne's test scores that cannot be supported by the data.

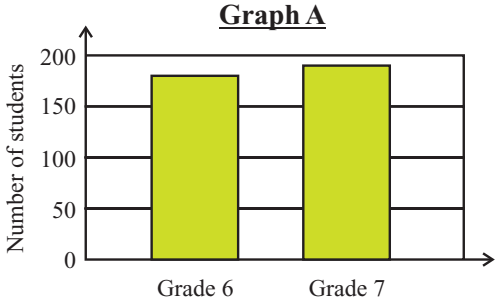
13. Write a claim about all three students' combined scores that can be supported by the data.

14. Write a claim about all three students' combined scores that cannot be supported by the data.

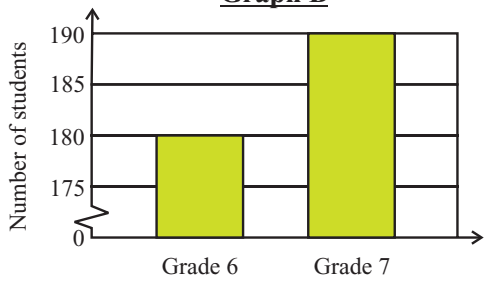
Lesson 5.5.2 — Evaluating Displays

The graphs below show the number of students in grades 6 and 7 who play sports.

Graph A



Graph B



1. Which graph makes it easier to find the exact number of students who play sports? Explain why.

2. What are the exact numbers of students that play sports in grades 6 and 7?

Winnie is looking at one of the graphs. She decides that the number of grade 7 students who play sports is double the number of grade 6 students who play sports.

3. Which graph is Winnie looking at? Explain your answer.

4. Is Winnie correct? If not, explain what mistake she has made.

5. Explain how changing the spacing on the horizontal axis of a line graph can change the appearance of the graph.

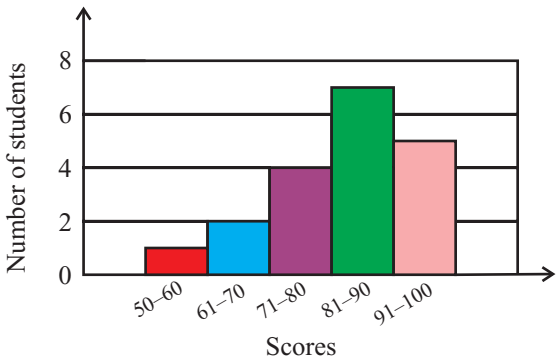
Mrs. Beckman made the graph on the right to display the test scores on last chapter's math test.

6. Give an advantage of grouping the test scores the way Mrs. Beckman did on her graph, instead of plotting each score separately.

7. Explain how the grouping of this data could be misleading.

8. Can you use this data display to determine the range of test scores? Explain your answer.

9. Can you use this data display to find the mode test score? Explain your answer.



Lesson 6.1.1 — Listing Possible Outcomes

1. What is the difference between an outcome and an event?

2. Can one outcome match more than one event? Explain your answer using an example.

For Exercises 3–5, sketch a spinner with 8 sections.

3. Add squares to the spinner sections so that 3 possible outcomes match the event "spin a square".

4. Add triangles to the spinner sections so that 2 possible outcomes match the event "spin a triangle".

5. Add another shape to the remaining sections. Write an event that matches these outcomes.

Mr. Sanchez is making 2 spinners for his students to use to determine the subject and type of project each student must complete.

He wants the subject spinner to have 3 events: spinning history, geography, or culture.

6. Sketch a spinner that matches the description of the subject spinner.

7. How many outcomes match the event "does not spin history"?

Mr. Sanchez wants the type of project spinner to have 4 events, spinning diorama, poem, art picture, or report.

8. Sketch a spinner that matches the description of the type of project spinner.

9. How many outcomes match the event "does not spin report"?

Each student will spin the subject spinner and the type of project spinner.

10. List all the outcomes of spinning both spinners that match the event "does not spin geography".

Lesson 6.1.2 — Tree Diagrams

1. Malcolm is using a tree diagram to find how many outcomes there are for spinning a spinner and rolling a die. Explain whether or not it matters which action he puts first in his tree diagram.

Elvin is decorating for the school dance. He can choose from the lists of decorations on the right.

Streamer Colors
Red
Yellow
Blue

Table Top Decorations
Rockets
Airplanes

Hanging Decorations
Stars
Moons
Planets
Clouds

2. Make a tree diagram that shows the number of combinations using one color of streamer, one type of hanging decoration and one type of table top decoration.

3. How many outcomes do not include Rockets as a table top decoration?

4. Elvin doesn't have enough clouds to hang over the last 10 tables at the dance. How many outcomes do not include clouds hanging over the table?

5. Elvin doesn't have enough blue streamers for the last four tables. How many outcomes do not include clouds or blue streamers?

Lesson 6.1.3 — Tables and Grids

Oliver is choosing from three different hats, and three different types of nose to put on his snowman. The types of hat are: top hat, cowboy hat, beret. The types of nose are: carrot, button, quarter.

1. Make a table to show all the combinations of one hat and one nose Oliver can make.

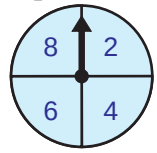
2. How many combinations match the event "with a button nose"?

3. List the combinations that match the event "with a cowboy hat".

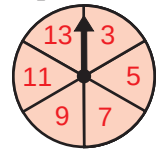
4. Oliver could not find a quarter. How many possible combinations of hats and noses are there now?

Exercises 5–9 are about the spinners shown on the right. Kayla spins Spinner A once, then spins Spinner B once, and then multiplies the two numbers together.

Spinner A



Spinner B



5. Make a table to show all the combinations of products Kayla can make with one spin on each spinner.

6. How many combinations match the event "product is an odd number"?

7. How many combinations match the event "product is an even number"?

8. Make a list of all the combinations that match the event "product greater than 60".

9. Make a list of all the combinations that match the event "product greater than 30 but less than 40".

Lesson 6.2.1 — Probability

Exercises 1–4 are about the set of cards shown below. The cards are turned over and shuffled, and then one card is picked. Find the probability of each of the following. Write your answer as a decimal.



1. Picking a card with an I

2. Picking a card with an F

3. Picking a card with an A

4. Picking a card with a J

5. List three ways the probability of an event occurring can be written.

6. Write two fractions that represent the same probability as 30%.

Tyrese wants to make a spinner with the following probabilities:

- 40% chance of spinning the number 1.
- 20% chance of spinning the number 4.
- 10% chance of spinning the number 8.
- 30% chance of spinning the number 2.

7. Draw a spinner that fits this description.

8. Tyrese is going to spin the spinner once. Find the probability that he spins a number greater than 9.

9. Find the probability that he spins a number greater than 3.

Lesson 6.2.2 — Expressing Probability

In Exercises 1–4 write each answer as a fraction in its simplest form.
Maggie has a box of 120 envelopes. It contains 40 red envelopes, 30 green envelopes and 50 yellow envelopes. She is going to pick one envelope out. Find:

1. P(Maggie picks a green envelope)
2. P(Maggie picks a red envelope)
3. P(Maggie picks a yellow envelope)
4. P(Maggie picks an envelope that is not yellow)

Tracie is playing a game where she mixes up 36 cards and picks one without looking.
Each card has a picture of a shape on it. There are 16 circles, 6 squares, 9 triangles, and 5 stars.
Write each of the following probabilities as a fraction in its simplest form:

5. P(Tracie picks a circle)
6. P(Tracie picks a square)
7. P(Tracie does not pick a square)

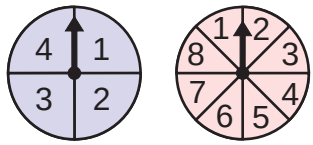
Manuel owns a pet store. He sells four types of bird feed. He currently has 144 bags of safflower seed, 24 bags of sunflower seed, 36 bags of cracked corn and 36 bags of Spanish peanuts. Manuel picks a bag of bird feed at random from his stock. Write each of the following probabilities as a percent:

8. P(picking sunflower)
9. P(picking safflower)
10. P(picking cracked corn)

Lesson 6.2.3 — Counting Outcomes

Sarah spins each of the spinners shown below and multiplies the two numbers she spins together.

1. Draw a table to show the possible outcomes.



Find the following probabilities, if x is the product of the two numbers spun.

2. P($x = 32$)
3. P(x is less than 10)
4. P(x is 20 or greater)
5. P(x is even)

When the probability of an event occuring is written as a fraction, what is represented by:

6. the numerator?
7. the denominator?

Eldrick is making a pattern out of tiles. For each of three spots he randomly chooses either a green, white or yellow tile. He can use the same color more than once.

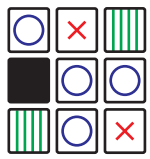
8. Draw a tree diagram to show the possible combinations of tiles Eldrick can choose.

Use your diagram from Exercise 8 to find the following probabilities.
Give each answer as a decimal to the nearest thousandth.

9. P (3 green tiles)
10. P (3 tiles of the same color)
11. P (exactly 2 tiles with the same color)
12. P (the 2nd and 3rd tiles being the same color)

Lesson 6.2.4 — Probability of an Event Not Happening

Find P(A) and P(not A) where event A is picking each of the following at random from the cards shown on the right. Write your answers as fractions.



1. a black card
2. a card with a circle
3. a striped card

4. The probability of event B happening is 20%. What is the sum of P (B) and P (not B)?

Lionel has 5 T-shirts and 4 pairs of shorts in his drawer.
This table shows the combinations of T-shirts and shorts he can choose.

Shorts \ T-Shirts	Orange	Purple	Striped	Dotted	Yellow
Blue	OB	PB	SB	DB	YB
White	OW	PW	SW	DW	YW
Grey	OG	PG	SG	DG	YG
Red	OR	PR	SR	DR	YR

Find P(C) and P(not C) where event C is Lionel choosing each of the following from the drawer. Give your answers as percents.

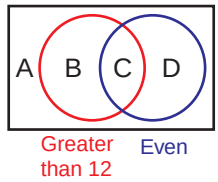
5. Striped shirt
6. Dotted shirt and red shorts

Lionel added a pair of tan shorts to the drawer.

7. What is the new total number of combinations?
8. Explain how your answer to Exercise 6 would change with the tan shorts included.
9. Find the probability that the tan shorts would not be pulled out of the drawer.

Lesson 6.2.5 — Venn Diagrams

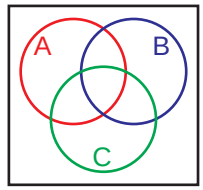
A number spinner is spun twice and the results multiplied.
In which section of the Venn Diagram does each of the following outcomes belong?



1. 4×4 **2.** 9×2 **3.** 2×6
4. 9×9 **5.** 9×1 **6.** 4×2

7. Nicole placed an outcome outside of the circles on a Venn Diagram.
What does this tell you about the outcome?
8. One circle on a Venn Diagram was placed entirely within another circle. What does this mean?

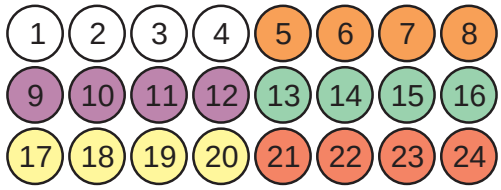
The Venn Diagram on the right shows three events for picking a number from 1 to 200 at random.
Event A is greater than 100. Event B is a multiple of 25. Event C is an even number.



9. Copy the diagram and place the numbers 1, 2, 25, 50, 100, 150, 175 and 200 in the correct sections of the diagram.
10. Name an outcome that would only match event A.
11. Name an outcome that would be placed in the overlapping area between circles A and C, but not in circle B.

Lesson 6.2.6 — Combining Events

Exercises 1–7 are about the set of 24 colored and numbered balls shown below.
One ball is picked at random from the set.
Calculate each probability and write your answers as percents.



1. P(green) **2.** P(green or yellow)
3. P(orange and greater than 6) **4.** P(red or less than 21)

Event A is "picking a number greater than 10 and less than 20,"
Event B is "picking a purple ball," and Event C is "picking an even number."
Find the following probabilities:

5. P(Event A and C) **6.** P(Event B or C) **7.** P(Event A and B and C)

8. Explain how P(A and B) is different from P(A or B).
9. Explain why you can't always just add P(A) to P(B) to finding P(A and B)

10. P(A) is 30% and P(B) is 20% with no overlap between events. What is P(neither A nor B)?

Lesson 6.3.1 — Independent and Dependent Events

Sierra and Noah are each choosing a number between 1 and 100. Say whether each of the following pairs of events for picking the numbers are dependent or independent. Explain your answers.

1. Event A: Sierra thinks of a number, but doesn't tell Noah what it is.
Event B: Noah thinks of a number, but doesn't tell Sierra what it is.
2. Event A: Sierra thinks of a number and tells Noah what she picked.
Event B: Noah thinks of a different number.
3. Event A: Sierra randomly picks one of 100 cards numbered 1–100 and keeps it.
Event B: Noah picks one of the remaining 99 cards.
4. Event A: Sierra randomly picks one of 100 cards numbered 1–100 and keeps it.
Event B: Noah randomly picks one of 100 balls numbered 1–100 out of a bag and keeps it.
5. Event A: Sierra randomly picks one of 100 balls numbered 1–100 out of a bag, then replaces it.
Event B: Noah randomly picks one of 100 balls numbered 1–100 out of the same bag.

6. A number cube is rolled, and a number of red counters equal to the number rolled are put into a box with 10 blue counters. Are the events "picking a red counter from the box" and "rolling 6 on the number cube" independent or dependent? Explain your answer.

Lesson 6.3.2 — Events and Probabilities



Charles has a box containing the 8 cards shown on the right.
He picks one card from the box, and then picks a second card without replacing the first. Find the following, writing your answers as decimals to 3 decimal places:

1. The probability that the first card picked is red
2. The probability that the first card picked is blue
3. The probability that the second card picked is blue, if the first was red
4. The probability that the second card picked is blue, if the first was blue
5. The probability that the first card picked is a circle
6. The probability that the first card picked is a square
7. The probability that the second card picked is a square, if the first was a circle
8. The probability that the second card picked is a square, if the first was a square
9. What would the answers to Exercises 3 and 4 be if Charles put the first card back in the box before he picked the second card?
10. Complete the following sentence:
The events "picking a red card on the first pick" and "picking a blue card on the second pick" are _____ if the first card is replaced and _____ if the first card is not replaced.

Lesson 6.3.3 — Calculating Probabilities of Independent Events

In Exercises 1–3, calculate P(A and B). A and B are independent events.

1. P(A) = 0.15; P(B) = 0.5 **2.** P(A) = 1; P(B) = 0 **3.** P(A) = 0.99; P(B) = 0.75

4. The probability that it will rain today is 20%. The probability that it will rain tomorrow is 30%. What is the probability that it will rain both today and tomorrow?

Each Monday a cooking teacher puts a star under one of six placemats.
Six students, three boys and three girls, come in and sit randomly in front of the placemats.
Calculate the probabilities of the following events:

5. Amy, a student in the class, will sit at the place with the star the first and second weeks.
6. A boy will sit at the place with the star the first and second week.
7. Amy will sit at the place with the star the first week and a boy will sit at it the second week.
8. Someone other than Amy will sit at the placemat the first week and a boy will sit at it the 2nd week.

Lesson 6.4.1 — Relative Frequency

Spinner A has the outcomes X and Y. Spinner B has outcomes 1, 2 and 3.
The two spinners were spun 12 times. The results are recorded on the table below.

Spinner A	Y	X	X	Y	X	Y	Y	X	Y	Y	Y	X
Spinner B	2	3	1	2	1	1	3	2	2	1	3	2

Calculate the relative frequency of the following events. Write your answers as fractions.

1. Spinner A landing on Y **2.** Spinner B landing on 3. **3.** Spinner B landing on 1 or 2.
4. Spinner B not landing on 1 **5.** Spinner A landing on Y and spinner B landing on 2

6. Explain the difference between theoretical probability and experimental probability.

The colors of 120 gumballs that came out of a gumball machine were recorded. The table on the right shows the number and relative frequency of the four colors of gumball. Write the relative frequencies of the following as fractions in their simplest form:

7. Blue gumballs **8.** Red gumballs
9. How many green gumballs were there?
10. How many yellow gumballs were there?

Color	Number of gumballs	Relative frequency
Red	26	
Green		$\frac{7}{24}$
Blue	31	
Yellow		$\frac{7}{30}$

Lesson 6.4.2 — Making Predictions

In Exercise 1–3, use the relative frequencies to predict how many times event A will happen in 50 trials.

1. P(A) = 0.8 **2.** P(A) = 4% **3.** P(A) = 34%

If the relative frequency of Event A is 0.15, predict how many the event will happen in:

4. 380 trials **5.** 640 trials **6.** 2000 trials

7. Describe the difference between biased and unbiased experiments.



One card from the set shown on the left was picked at random then replaced.

A	B	C	D	E	F
4	2	5	6	3	5

The results of doing this 25 times are shown in the table on the left.

8. Find the experimental probability of each card being pulled.

9. How would you expect the experimental probabilities you calculated in Exercise 8 to be affected if the number of trials was increased to 1000?

Lesson 7.1.1 — Parts of a Circle

In Exercises 1–4, choose the correct definition from the table below to match each word.

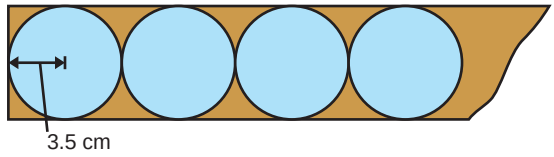
1. Diameter
2. Circle
3. Radius
4. Center

A. The set of all points in a plane that are all the same distance from the center
B. The distance from the center to the edge of a circle
C. The point at the middle of a circle that is an equal distance from all points on the circle
D. The distance from one edge of a circle to the other in a straight line through the center

5. The radius of a circle is y inches. Write an expression that could be used to determine the diameter of the circle.

Harry is placing equally-sized water bottles into a box as shown in the diagram below.

6. How long is the shortest side of the box?



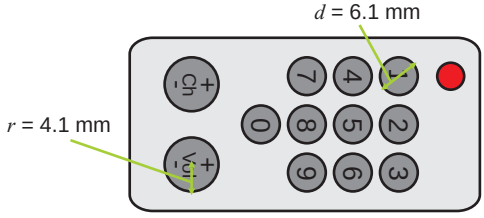
Harry wants to know how many bottles will fit into a box. In Exercises 7–9, find how many water bottles would fit into a box with the following lengths:

7. 84 centimeters **8.** 30 centimeters **9.** 68 centimeters

Lesson 7.1.2 — Circumference and π

Exercises 1–3 are about the remote control shown in the picture. Use 3.14 as π. Find the circumference of:

1. Button number 1 **2.** The volume button.



3. The circumference of the red button is 12.25 mm. What is its diameter?

4. Kyle says that the value of π changes in every equation. Is he correct? Explain your answer.

5. The diameter of a large circle was 10 times as long as the diameter of a smaller circle. How many times greater is the circumference of the large circle?

A cylindrical container is used to send items back and forth to a drive thru bank teller.

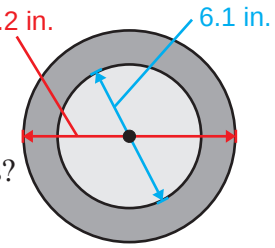
6. The radius of the container's circular lid is 4.3 inches. Find the circumference of the lid.

The container passes through a tube. The diameter of the tube is 2 inches greater than the diameter of the container.

7. Calculate the circumference of a circle with the same diameter as the tube.

Lesson 7.1.3 — Area of a Circle

Use 3.14 for π in Exercises 1–7.



The diagram on the right is from a poster showing a car wheel.

1. Find the approximate area of the smaller circle.
2. Find the approximate area of the larger circle.
3. What is the approximate area of the part of the diagram between the two circles?

Marlene has a rectangular piece of cardboard measuring 16 inches by 8 inches.
She wants to cut out 2 circles each with a diameter of 7 inches from the cardboard.

4. Calculate the approximate area of each circle.
5. Find the approximate area of the cardboard that is left after the circles have been cut out.



Marcie is making a circular spinner like the one shown in the diagram.
The spinner has a radius of 6.25 centimeters.

6. Find the approximate area of the spinner. Round your answer to the nearest tenth.
7. Find the approximate area of the spinner that will be colored yellow.

Lesson 7.2.1 — Describing Angles

1. Copy and complete the following sentence:
All right angles have measures _____ than acute angles but _____ than obtuse angles.

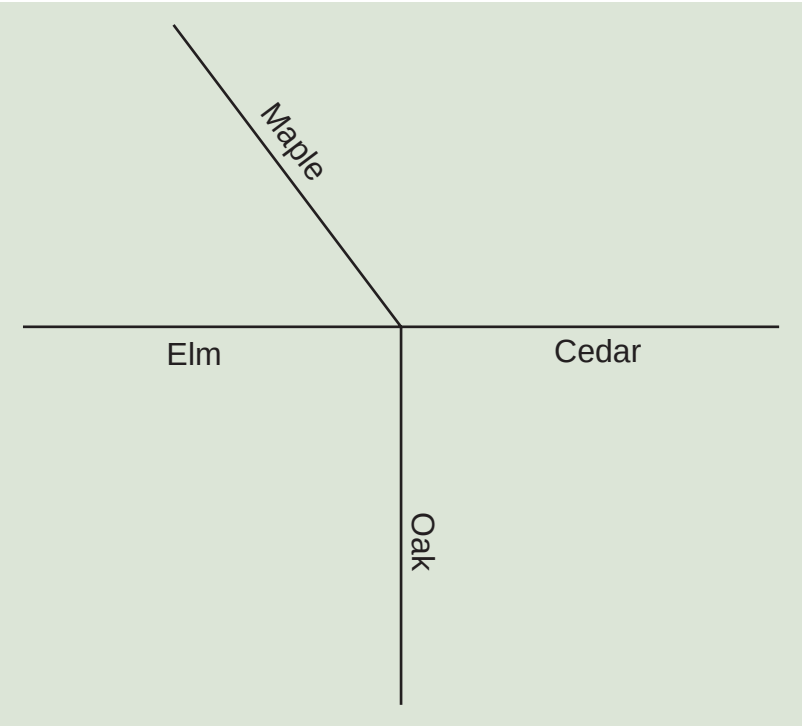
The map on the right shows some of the road in Jamal's hometown.

2. Which two pairs of roads appear to meet at right angles on the map? Measure the angles between them to check this.

3. Which pair of roads appear to meet at an acute angle on the map? Measure the angle that they create.

4. Which two pairs of roads appear to meet at obtuse angles on the map? Measure the angles that they create.

5. Birch Road is under construction and will divide the angle created by Maple and Cedar exactly in half. What will be the measure of the angle created by Birch and Maple road?

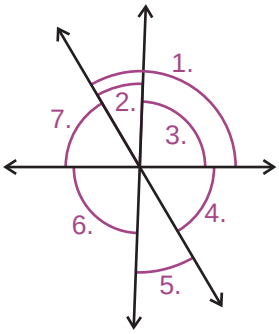


In Exercise 6–9, choose the correct definition from the table below to match each word.

6. Angle
7. Protractor
8. Ray
9. Vertex

A. A part of a straight line that has one end point and goes on forever in one direction
B. A point where the two sides of an angle meet
C. A figure made when two rays or two line segments meet at the same end point
D. An instrument used to measure angles in degrees

Lesson 7.2.2 — Pairs of Angles



Use the diagram on the left to answer Exercises 1–9. Say whether the following pairs of angles are adjacent, linear, vertical, or none of these.

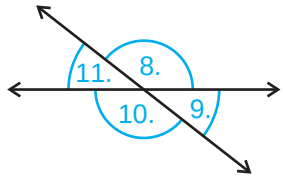
1. Angle 1 and angle 2
2. Angle 1 and angle 7
3. Angle 2 and angle 5
4. Angle 2 and angle 4
5. Angle 6 and angle 7
6. Angle 3 and angle 6

7. Angle 1 measures 121°. What is the measure of angle 4?

If angle 3 measures 87°, find the measures of:

8. Angle 6
9. Angle 5

In Exercises 10–12 use the diagram on the right.
Steve used chalk to draw the two lines shown on the sidewalk.



10. Name 2 angles that appear to be vertical angles in the sketch.
11. Name 2 angles that appear to be linear angles on the sketch.
12. Jake then added another line to the sketch that divided angle 8 exactly in half. If angle 10 measures 142°, what is the measure of half of angle 8? Explain your answer.

Lesson 7.2.3 — Supplementary Angles

1. Explain how to determine if two angles are supplementary.

2. Marsha said that all linear pairs must be supplementary. Is this true? Explain your answer.

3. Can two angles be supplementary if one is acute and one is right? Explain your answer.

Kendra and Alejandro are cutting up a circular pie. The pie is cut exactly in half in a straight line, and they are each given exactly half of the pie to cut into slices.

4. What is the angle measure of each half of the pie?

5. From her half, Kendra cuts one slice of pie with an angle measure of 72º, and one with an angle measure of 45°. What is the measure of the angle made by Kendra's remaining piece of pie?

6. From the second half of the pie, Alejandro cuts one slice with an angle of 68º, then cuts the remaining piece exactly in half. What is the measure of the angle made by each of the two slices Alejandro makes with his second cut?

Lesson 7.2.4 — The Triangle Sum

The angle measures of three triangles were recorded on the table shown on the right. Find:

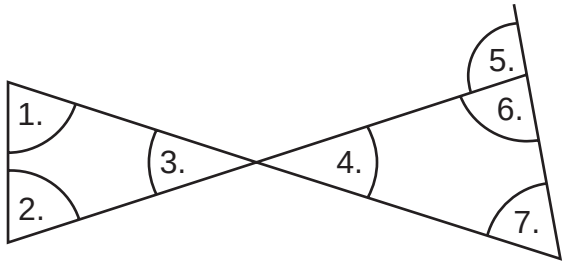
1. x
2. y
3. z

Triangle	Angle 1	Angle 2	Angle 3
A	50°	60°	x°
B	y°	31°	19°
C	6°	z°	108°

4. Explain how knowing the measure of a supplementary angle to an angle in a triangle can help determine the measure of the other angles in a triangle.

In this diagram, the measures of angles 1 and 2 are equal. The measure of angle 3 is 35°, and angle 5 measures 109°. Find the measure of:

5. Angle 1
6. Angle 2
7. Angle 6
8. Angle 7



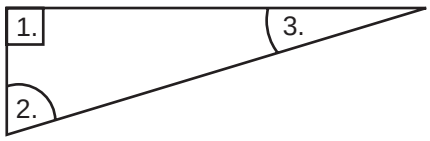
Lesson 7.2.5 — Complementary Angles

1. Copy the following sentence and fill in the missing words:
_____ angles add up to 90º but _____ angles add up to 180º.

2. If one of the angles in a triangle is a right angle, what does that tell you about the other two angles?

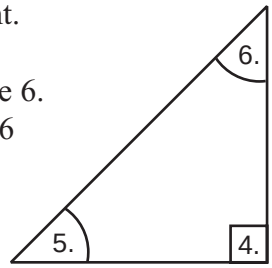
3. Will found that two angles in a triangle are complementary. What is the measure of the third angle?

In this diagram, angle 2 measures 12º less than angle 1.



4. Find the measure of angle 2.
5. Find the measure of angle 3.

The top view of a box used to hold chocolates is shown in the diagram on the right. The triangle is an isosceles triangle.



6. Find the measure of angle 5.

7. Find the measure of angle 6.

The company that makes the box decides to change the measures of angles 5 and 6 so that they are still complementary, but not equal.
The new measure of angle 5 will be 52.5º.

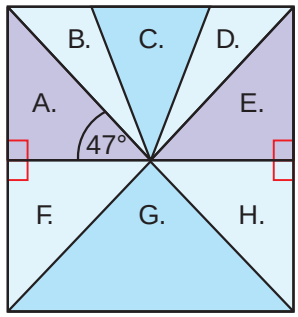
8. Find the measure of angle 6 in the new triangle.

Lesson 7.3.1 — Classifying Triangles by Angles

In Exercises 1–4, say whether each sentence is true or false, and explain your answer.

1. In an obtuse triangle, two of the angles must be obtuse.
2. In a right triangle, only one angle can be a right angle.
3. An acute triangle must have 3 angles that are less than 90º.
4. All the angles in an obtuse triangle have greater measures than all the angles in an acute triangle.

A design for a stained glass window is shown in the diagram.



5. Find the measure of the missing angle in triangle A.
6. Is triangle A angle right, obtuse or acute?
7. Name two triangles in the picture that appear to be obtuse.
8. If the largest angle in triangle G was 3° larger, it would be a right angle. What is the measure of this angle?
9. The other angles in triangle G are the same size. Find their measures.
10. Is triangle G right, obtuse or acute?

Lesson 7.3.2 — Classifying Triangles by Side Lengths

This table has been used to record the side lengths of seven triangles.

Triangle	Side 1	Side 2	Side 3	Type
A	9 mm	10 mm	9 mm	
B	5.86 m	12 m	5.9 m	
C	9.2 ft	9.2 ft	9.2 ft	
D	1.3 in	3.5 in	1.3 in	
E	4.5 cm		5.5 cm	Scalene
F	6 cm	9 cm		Isosceles
G	2 cm	2 cm		Equilateral

Are the following equilateral, isosceles or scalene?

1. Triangle A
2. Triangle B
3. Triangle C
4. Triangle D

5. The missing lengths from the last three rows of the table are 2 cm, 3 cm and 6 cm. Match each of these missing length to the correct triangle.

6. Linda says that a right triangle could not be an isosceles triangle. Is this true? Explain your answer.

7. Luis says that if you flip an equilateral, isosceles or scalene triangle over to get a mirror image, the new triangle will be the same type of triangle as the original. Is this true? Explain your answer.

Lesson 7.3.3 — Types of Quadrilaterals

Quadrilateral	Angle 1	Angle 2	Angle 3	Angle 4
1.	50°	55°	226°	$a°$
2.	115°	110°	76°	$b°$
3.	82°	98°	$c°$	82°
4.	$d°$	74°	106°	$e°$
5.	35°	35°	$f°$	$g°$

This table shows the angle measures of five quadrilaterals. Use it to answer Exercises 1–6.

1. Find a **2.** Find b

3. Find c

4. One of quadrilaterals 1, 2 and 3 is an isosceles trapezoid. Which one is it? Explain you answer.

5. Quadrilateral 4 is an isosceles trapezoid. If $d > e$, find d and e.

6. Quadrilateral 5 is a parallelogram. Find f and g.

In Exercises 7–9, use the words below to complete each sentence. Use each word only once.

Square Rhombus Trapezoid Parallelogram Rectangle

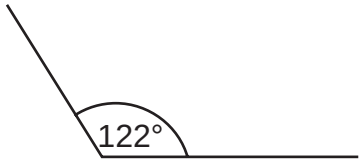
7. A _____ has 4 sides of equal length, but does not have to have 4 equal angles.

8. A _____ is a _____ with 4 sides the same length.

9. A square is not a _____, but is a _____.

Lesson 7.3.4 — Drawing Quadrilaterals

Andrew started drawing a parallelogram as shown on the right.
The two sides he has drawn are the correct length.



1. What is the next step Andrew should take to draw the parallelogram?

2. Find the measure of the next angle Andrew will need to draw.

Reggie is making a greeting card for his mom in the shape of an isosceles trapezoid.

3. One angle on the trapezoid must be 130°. Write the measures of the other 3 angles in the trapezoid.

4. The bottom base of the trapezoidal card will be 2 times as long as the top base.
Suggest 2 possible measures for the bases.

5. Draw the trapezoid described. Label the angles and the base lengths on the trapezoid you drew.

6. Describe how to draw a rhombus when only 1 side length and 1 angle is known.

Lesson 7.4.1 — Three Dimensional Figures

In Exercise 1–4 say whether each sentence is true or false, and explain your answer.

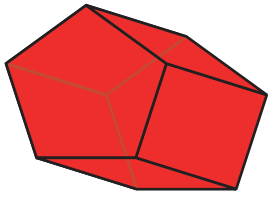
1. The number of edges on a rectangular prism is always 4 more than the number of vertices.

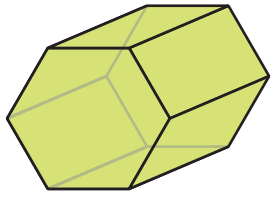
2. A cube and a triangular prism each have 6 faces.

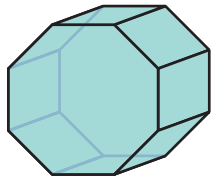
3. A cylinder has 2 circular faces.

4. A triangular prism has 3 fewer edges than a rectangular prism.

Exercises 5–10 are about the prisms shown below.

A. 

B. 

C. 

For each of the prisms shown, find the number of:

5. faces **6.** edges **7.** vertices

If the base of a prism is a polygon with n vertices, find an expression for:

8. The number of faces in the prism **9.** The number of edges in the prism

10. The number of vertices in the prism

Lesson 7.4.2 — Volume of Rectangular Prisms

In Exercises 1–4 choose a reasonable estimate of the volume of each object.
Choose from: 100 cm^3, 1000 cm^3, 10,000 cm^3, or 100,000 cm^3

1. Volume of a toybox

2. Volume of a shoe box

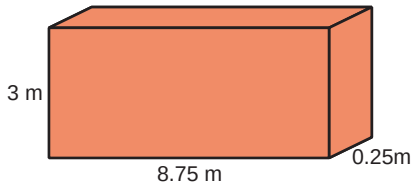
3. Volume of a slice of bread

4. Volume of a small loaf of bread

5. List the dimensions of a cube that has a volume of 1 cubic yard.

6. Describe how the volume of a cube changes when 3 cubes of the same size are added to it.

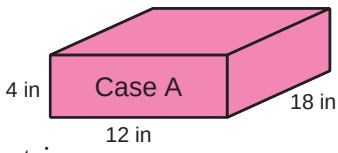
A construction worker built a wall with dimensions as shown in the picture.

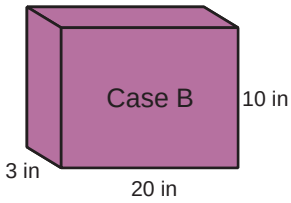


7. Find the volume of the wall.

8. Find the volume of the wall if the thickness was increased from 0.25 meters to 0.5 meters.

Josie had these 2 suitcases available for a trip.





9. Find the volume of Suitcase A.

10. Find the volume of Suitcase B.

11. Find the difference in these volumes.

Josie took the suitcase with the bigger volume on her trip.

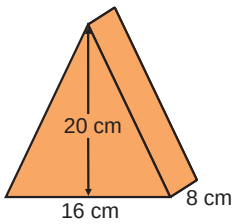
She only filled it $\frac{3}{4}$ of the way full.

12. Find the volume of empty space left in the suitcase.

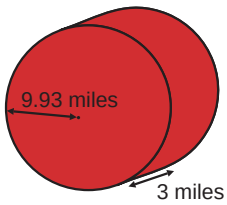
Lesson 7.4.3 — Volume of Triangular Prisms and Cylinders

In Exercises 1–4, calculate the volume of each figure. Use 3.14 for π.

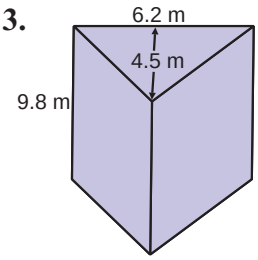
1.



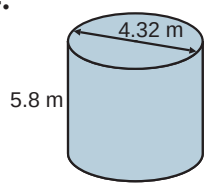
2.



3.



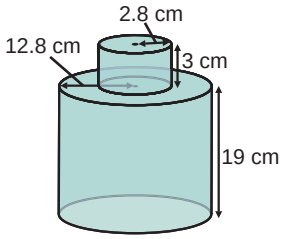
4.



5. Explain how the volume of a rectangular prism would change if the prism was divided in half.

6. Explain how to find the height of a triangular prism if you know the volume of the prism and the area of its base.

In Exercises 7–9, use 3.14 for π.
The lid on water bottle is pictured in the diagram.



7. Find the volume of the lid.

8. Find the volume of the bottle.

9. Find the difference between these volumes.

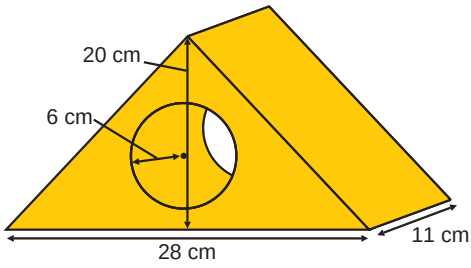
10. The volume of a triangular prism is 3.5 cm^3 and the area of the base is 7.0 cm^2.
What is the height of the prism?

11. A cylinder has a volume of 366 in^3. Find the height of the cylinder if its base has a radius of 4 in.
Use 3.14 for π.

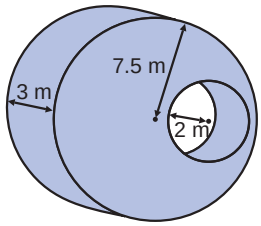
Lesson 7.4.4 — Volume of Compound Solids

Each of the figures below has a hole cut out of them in the shape of a cylinder.
Determine the volume of each figure after the hole was cut out. Use 3.14 for π.

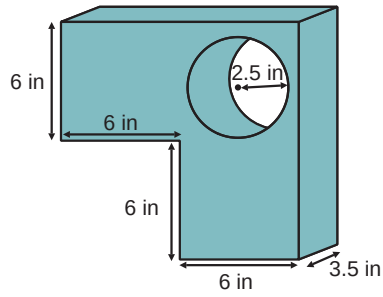
1.



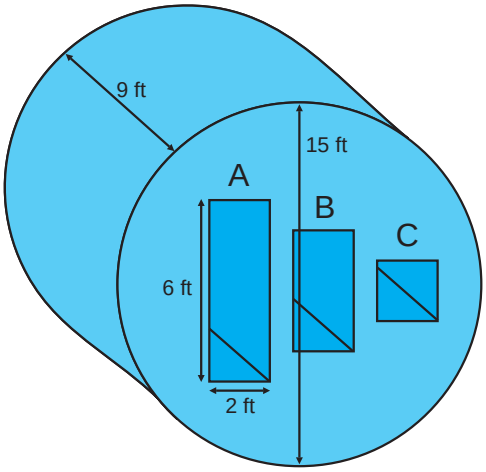
2.



3.



4. Describe how the volume of a 3D figure would change if a cube was cut out of the middle of it.



In Exercises 5–8, use the diagram on the left.
Boxes A, B and C all extend all the way through the cylinder.
Allison is practicing pitching. She first tried to pitch into box A as shown in the figure.

5. Find the volume of box A.

Next, she pitched into box B. The volume of box B is $\frac{2}{3}$ of the volume of box A.

6. Find the volume of box B.

Then, Allison pitched into box C.
The volume of box B is twice as much as the volume of box C.

7. Find the volume of box C.

8. Find the volume of the cylinder that was left after the 3 boxes were cut out of it. Use 3.14 for π.

Lesson 7.5.1 — Generalizing Results

1. What does the letter n in the expression $n + 6$ represent?

2. Explain why it is important to test a generalization.

This table shows information about the faces, vertices and edges of prisms.

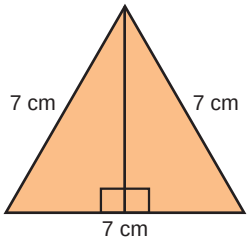
3D Figure	Number of faces	Number of vertices	Number of faces and vertices	Number of edges
Triangular Prism	5	6	11	9
Rectangular Prism	6	8	14	12
Pentagonal Prism	7	10	17	15
Hexagonal Prism	8	12	20	18
Heptagonal Prism	9	14	p	q
Octagonal Prism	10	16	r	s

3. Determine the value of p.
4. Determine the value of q.
5. Determine the value of r.
6. Determine the value of s.

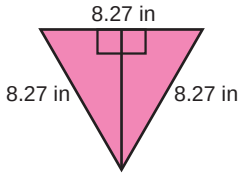
7. Write how you could predict the number of edges on a prism if the number of faces was 100 and the number of vertices was 196.

8. Write an equation to find the number of edges on any prism.
Use v for the number of vertices, f for the number of faces and e for the number of edges.

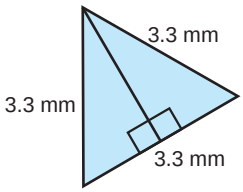
In Exercises 1–4 use the diagram below, showing four equilateral triangles.



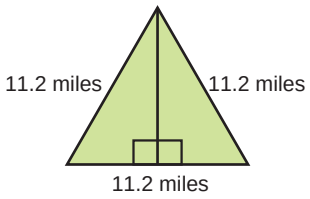
Area = 21.2 cm^2



Area = 29.6 in^2



Area = 4.7 mm^2



Area = 54.3 $miles^2$

Using the equilateral triangles shown above, decide whether the statements in Exercises 1–4 appear to be true or false. If any are false, give an example that proves this.

1. All equilateral triangles have 3 sides the same length.

2. If a line is drawn from any vertex of an equilateral triangle to the midpoint of the side opposite it, 2 right angles are created at the midpoint.

3. The length of any side on an equilateral triangle is always equal to half the perimeter of the triangle.

4. The area of an equilateral triangle is always half as much as the perimeter of the triangle.

5. The area of an equilateral triangle is always more than 3 times the side length of the triangle.

6. Draw several right triangles and use your sketches to decide which of the statements about equilateral triangles from Exercises 1–5 are true for right triangles, and which are false.

7. Andrew said that a total of 2 examples are needed to prove a statement is true. Is Andrew correct? Explain your answer.

Appendixes

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Formula Sheet 473
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Glossary

Symbols

$<$	is less than	$\neq$	is not equal to
$>$	is greater than	$\mathbb{N}$	the *natural numbers*
$\leq$	is less than or equal to	$\mathbb{W}$	the *whole numbers*
$\geq$	is greater than or equal to	$\mathbb{Z}$	the *integers*

A

adjacent angles two angles that share a common side and a vertex

angle the amount of turn between two straight lines meeting at a point; there are 360° in a full turn

arc part of a circle's circumference; can be drawn with a compass

associative properties (of addition and multiplication) for any a, b, c: $a + (b + c) = (a + b) + c$
$a(bc) = (ab)c$

B

base in the expression b^x, the base is b

bias unfair influence on a result

C

central tendency the value of "typical" items in a data set

circumference the distance around the outside of a circle

common factor a number or expression that is a *factor* of two or more other numbers or expressions

common multiple a multiple of two or more different integers

commutative properties (of addition and multiplication) for any a, b: $a + b = b + a$ and $ab = ba$

complementary angles two angles whose measures sum to 90°

conversion factor the ratio of one unit to another; used for converting between units

cube a three-dimensional figure with six identical square faces

customary units the system of units that includes: inches, feet, yards, miles, ounces, and pounds

cylinder a three-dimensional figure with two circular bases and a constant cross-section

D

data set a collection of information, often numbers

decimal a number including a decimal point; digits to the right of the decimal point show parts of a whole number

denominator the bottom expression of a fraction

diameter a straight line from one side of a circle to the other, passing through the center

distributive property (of multiplication over addition) for any a, b, c: $a(b + c) = ab + ac$

E

edge on a three-dimensional figure, an edge is where two faces meet

equation a mathematical statement showing that two quantities are equal

equilateral triangle a triangle whose sides are all the same length

equivalent fractions fractions are equivalent if they have the same value

estimate an inexact judgement about the size of a quantity; an "educated guess"

evaluate find the value of an expression by substituting actual values for variables

event a description that matches one or more possible outcomes

exponent in the expression b^x, the exponent is x

expression a collection of numbers, variables, and symbols that represent a quantity

F

face a flat surface of a three-dimensional figure

factor a number or expression that can be multiplied to get another number or expression — for example, 2 is a factor of 6, because $2 \times 3 = 6$

frequency the number of times that something happens

G

greatest common factor (GCF) the largest expression that is a *common factor* of two or more other expressions; all other common factors will also be *factors* of the GCF

grouping symbols symbols that show the order in which mathematical operations should be carried out — such as parentheses and brackets

I

improper fraction a fraction whose numerator is greater than its denominator — for example $\frac{7}{4}$

independent events events whose probabilities are not affected by whether the other happens or not

integers the numbers 0, ±1, ±2, ±3,...;
the set of all integers is denoted $\mathbb{Z}$

interest extra money you pay back when you borrow money, or that you receive when you invest money

inverses a number's *additive inverse* is the number that can be added to it to give 0;
a number's *multiplicative inverse* is the number that it can be multiplied by to give 1

isosceles triangle a triangle with two sides of equal length

L

least common multiple (LCM) the smallest integer that has two or more other integers as *factors*

M

manipulate to change an expression or equation

Glossary (continued)

mean a measure of central tendency; the sum of a set of values, divided by the number of values in the set

median the middle value when a set of values is put in order

metric the system of units that includes: centimeters, meters, kilometers, grams, kilograms, and liters

mixed number a number containing a whole number part and a fraction part

mode the value that occurs most frequently in a set

N

natural numbers the set of numbers 1, 2, 3,...; the set of all natural numbers is denoted $\mathbb{N}$

numerator the top expression of a fraction

numeric expression a number or an expression containing only numbers and operations (and therefore no *variables*)

O

origin on a number line, the origin is at zero

outcome a possible result of an experiment

P

parallelogram a four-sided shape with two pairs of parallel sides

percent value followed by the % sign; corresponds to the numerator of a fraction with 100 as the denominator

perimeter the sum of the side lengths of a polygon

population the full group that samples can be taken from

power an expression of the form b^x, made up of a base (b) and an exponent (x)

prime factorization a factorization of a number where each *factor* is a *prime number*, for example $12 = 2 \times 2 \times 3$

prime number a natural number that can only be divided by itself and 1, with exactly 2 factors

prism a three-dimensional figure with two identical bases and a constant cross-section

probability a number which shows how likely an event is to happen, written as a number from 0 to 1, a percent, or a fraction

product the result of multiplying numbers or expressions together

proportion an equation showing that two ratios are equivalent

Q

quadrilateral a two-dimensional figure with four straight sides

quotient the result of dividing two numbers or expressions

R

radius the distance between a point on a circle and the center of the circle

range the difference between the lowest and highest values in a data set

rate a kind of ratio with units

ratio the amount of one thing compared with the amount of another thing

reciprocal the multiplicative *inverse* of an expression

relative frequency the number of times an outcome occurred, divided by the number of trials

rhombus a two-dimensional figure with four equal-length sides in two parallel pairs

rounding replacing one number with another number that's easier to work with; used to give an approximation of a solution

S

sample a small part of the population chosen to represent the whole population

sampling
in *convenience sampling*, each person or object is chosen because they are easy to access;
in *random sampling*, each person or object in the population has an equal chance of being selected;
in *systematic sampling*, people or objects are chosen by a set pattern;
self-selected sampling relies on people choosing to give information

sampling error a measure of how accurately a sample represents the population;
a sampling error for the mean is the difference between the mean of the sample and the mean of the population

scalene triangle a triangle with three unequal sides

sign of a number whether a number is positive or negative

similar two figures are similar if all of their corresponding sides are in proportion and all of their angles are equal

simplify to reduce an expression to the least number of terms, or to reduce a fraction to its lowest terms

solve to manipulate an equation to find out the value of a variable

sum the result of adding numbers or expressions together

supplementary angles two angles whose measures sum to 180°

survey information collected by contacting people by phone, mail, or in person and asking them to provide some data

T

terms the parts that are added or subtracted to form an expression

trapezoid a four-sided shape with exactly one pair of parallel sides

V

variable a letter that is used to represent an unknown number

Venn diagram a diagram showing how different items in sets are related, making probabilities easier to visualize

vertex the point on an angle where the two rays meet on a two-dimensional figure; the point where three or more faces meet on a three-dimensional figure

vertical angles angles that are opposite each other when two lines cross

volume a measure of the amount of space inside a three-dimensional figure

W

whole numbers the set of numbers 0, 1, 2, 3,...; the set of all whole numbers is denoted $\mathbb{W}$

Formula Sheet

Order of Operations

Perform operations in the following order:

1. Anything in **grouping symbols** — working from the innermost grouping symbols to the outermost.
2. **Exponents**.
3. **Multiplications** and **divisions**, working from left to right.
4. **Additions** and **subtractions**, again from left to right.

Fractions

Adding and subtracting fractions with the same denominator: $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$

Adding and subtracting fractions with different denominators: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$

Multiplying fractions: $a\cdot\frac{c}{d}=\frac{ac}{d}$ and $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$

Dividing fractions: $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$

Reciprocals: $\frac{d}{c}$ is the called reciprocal of $\frac{c}{d}$

Rules for Multiplying

positive × positive = positive
positive × negative = negative
negative × positive = negative
negative × negative = positive

Rules for Dividing

positive ÷ positive = positive
positive ÷ negative = negative
negative ÷ positive = negative
negative ÷ negative = positive

Axioms of the Real Number System

For any real numbers a, b, and c, the following properties hold:

Property Name	Addition	Multiplication
Commutative Property:	$a + b = b + a$	$a \times b = b \times a$
Associative Property:	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Distributive Property of Multiplication over Addition:	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$	

Applications Formulas

Investments

The return (I) earned in t years when p is invested at an interest rate of r (expressed as a fraction or decimal): $I = prt$

Speed

$\text{speed}=\frac{\text{distance}}{\text{time}}$ $\text{distance} = \text{speed} \times \text{time}$ $\text{time}=\frac{\text{distance}}{\text{speed}}$

Formula Sheet (continued)

Units

Lengths in Customary Units

1 foot (ft) = 12 inches (in.)
1 yard (yd) = 3 feet (ft)
1 mile (mi) = 1760 yards (yd)

Lengths in Metric Units

1 dekameter (dam) = 10 meters (m)
1 hectometer (hm) = 100 meters
1 kilometer (km) = 1000 meters
1 decimeter (dm) = 0.1 meters
1 centimeter (cm) = 0.01 meters
1 millimeter (mm) = 0.001 meters

Customary to Metric Conversions

1 inch (in.) = 2.54 cm
1 foot (ft) = 30.48 cm
1 yard (yd) = 0.91 m
1 mile (mi) = 1.61 km

Metric to Customary Conversions

1 cm = 0.39 inches (in.)
1 cm = 0.033 feet (ft)
1 m = 1.09 yards (yd)
1 km = 0.62 miles (mi)

Converting Between Temperatures in Fahrenheit and Celsius

$F = 1.8C + 32$
$C = (F - 32) \div 1.8$

Area

Area of a rectangle: $A = bh$

Area of a triangle: $A = \frac{1}{2}bh$

where b stands for the length of the base and h stands for the height.

Volume

$V = Bh$

where B represents the area of one of the prism's bases, and h stands for the height of the prism.

Cubes

Surface area of a cube: $A = 6s^2$
Volume of a cube: $V = s^3$

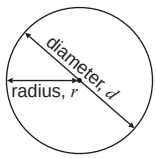
where s represents the length of the side of the faces.

Ratios

The ratio "a to b" can also be written: $\frac{a}{b}$ or $a : b$

Circles

Diameter: $d = 2r$
Circumference: $C = \pi d$
Area: $A = \pi r^2$



Probability

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

P(not A) = 1 – P(A)
P(A or B) = P(A) + P(B) – P(A and B)
P(A and B) = P(A) × P(B), for independent events

Statistics

For any numerical data set:

$$\text{Mean} = \frac{\text{sum of values}}{\text{total number of values}}$$

Median = middle value if the values are ordered
Mode = value that occurs most often
Range = maximum value – minimum value

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